Saint Vincent and the Logarithm

Gregorius Saint Vincent (1584–1667) was born in Belgium, studied theology, philosophy, and mathematics in Rome with Christoph Clavius (1537–1612), and was ordained a Jesuit priest in 1613. After Galileo announced his telescopic discoveries in his *Siderius nuncius* (1610), Saint Vincent hinted that he supported the heliocentric system.[1] This made him suspect to his Catholic superiors, and so he was forced to teach Greek and mathematics in various cities around Europe. His most famous work, *Opus geometricum quadraturae circuli et sectionum coni* (Geometrical works on the quadature of the circle and the conic sections) was written in the 1620s, but the Jesuits refused to let him publish it then. When he fled Prague in 1631 just ahead of the advancing Swedes, he was forced to leave the manuscript behind. He did not see it again until 1641. Finally, with the help of several students, it was published in 1647.

The volume is huge—containing more than 1250 large pages. It contains the first presentation of the summation of infinite geometric series, a method of trisecting angles using infinite series, and the result Saint Vincent considered his most important: a method for squaring the circle. Unfortunately, this result was incorrect, as Huygens first pointed out in 1651. Although this error destroyed his reputation, the work contains much of value which influenced Leibniz, Wallace, and Wren.

The frontispiece of the *Opus geometricum* is, without a doubt, the most magnificent allegory in all of mathematical publishing. In the foreground, Archimedes, who was killed by a Roman soldier in 212 B.C. during the sack of Syracuse, is drawing the diagram for his proof of the area of a circle[2]. Cowering attentively behind him is Euclid, who is looking on in awe. The character anachronistically wearing swim goggles[3] has not been identified. Wading in the estuary is Neptune, whose banner carries the slogan "Plus ultra," there is *more beyond* this ancient geometry, yet the ancients are prevented from getting there by the Pillars of Hercules.[4] But Gregorius has discovered this new land of mathematics—at least,

[1] This is the most exciting scientific book ever written. I encourage you to read the *Sidereus Nuncius or the Sidereal Messenger by Galileo Galilei*, translated and edited by Albert Van Helden, University of Chicago Press, 1989.

[2] *Measurement of a Circle*, Proposition I: "The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle." See Heath, *The Works of Archimedes* (1897), p. 91; reprinted by Dover.

[3] Eyeglasses were invented in 1285 by Allesandro de Spina.

[4] "In the word histories section [of the *Random House English Language Desk Reference*], there is a nice background to the verb *non plus*, which means "to make utterly perplexed"; it goes, "The original Latin phrase was 'non plus ultra,' meaning 'no more beyond' and allegedly inscribed on the *Pillars of Hercules*, beyond which no ship could safely sail." ...the "pillars" are the rocky promontories flanking the Strait of Gibralter, and as we

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this frontispiece claims so. In the background the sunbeam carries the words "Mutat quadrata rotundis" (the square is changed into a circle) which are illustrated by the putto holding the square frame which focuses the sunbeam into a circle on the ground. Note that the putti are tracing it out with a compass, and that the circle is correctly drawn in perspective as an ellipse[5].

The copy of the title page and frontispiece of the *Opus geometricum* which are reproduced here come from the copy in the United States Military Academy Library at West Point, which contains a very extensive collection of older mathematical works (a bibliography of which is under preparation).[6]

The most important aspect of this book for the calculus is a surprising connection between the natural logarithm and the rectangular hyperbola, $xy = 1$. Let $A_{a,b}$ denote the area above the interval $[a, b]$ and below the hyperbola $y = 1/x$. We prefer not to use integrals to describe this area so as not to prejudge the issue; indeed we are talking about some proto-calculus that was done before the time of Newton and Leibniz. Now let x_i, $i = 1, 2, \ldots, n$, partition the interval $[a, b]$ into equal pieces. Then Gregorius bounded the area by

know, there is nothing beyond there except the Atlantic Ocean and the place where you fall off the earth." So wrote William Saffire in "On Language: Gifts of Gab," *New York Times Magazine*, December 3, 1995, p. 38. Alas, the dictionary is wrong about people believing that one could fall off the earth. The flat earth myth was created by Washington Irving (1783–1859) in his romantic biography *History of the Life and Voyages of Christopher Columbus* (1828). See my "How Columbus encountered America," Mathematics Magazine, 65 (1992), 219–225.

5 The interpretation of this engraving is primarily my own. The only description of this frontispiece that I am aware of is "A curious mathematical title-page," by Florian Cajori, *The Scientific Monthly*, 14 (1922), 293–295.

6 Note that the work is signed by René François de Sluse (1622–1685), who developed a method for finding tangents to algebraic curves just before Newton (1642–1727) discovered his own. The volume also contains notes which, I conjecture, were written by Sluse.

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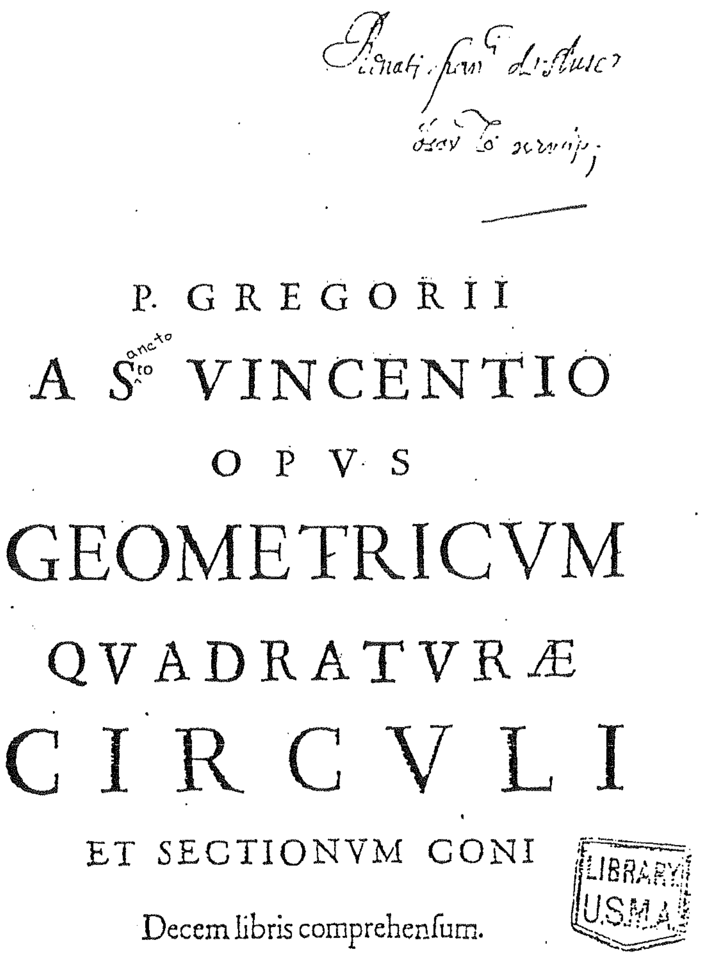
lower and upper "Riemann" (1826–1866) sums:

$$\sum_{i=1}^{n} \frac{b-a}{nx_i} \leq A_{a,b} \leq \sum_{i=1}^{n} \frac{b-a}{nx_{i-1}}$$

If we do the same for the interval $[ta, tb]$, partitioning it at the points tx_i, $i = 1, 2, \ldots, n$, then we obtain, in a similar fashion, the sums:

$$\sum_{i=1}^{n} \frac{tb-ta}{ntx_i} \leq A_{ta,tb} \leq \sum_{i=1}^{n} \frac{tb-ta}{ntx_{i-1}}$$

It is apparent that the ts cancel in these sums. Since both $A_{a,b}$ and $A_{ta,tb}$ are bounded by the same sums, they must be equal. Today we would use a limiting process here, but that was not Saint


P. GREGORII
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Vincent's way; he used a proof by exhaustion—to use a phrase that he coined in this book—although the method goes back to Archimedes.

A student of Gregorius, Alfonzo Antonio de Sarasa noted[7] that area is additive and if we put $a = 1$ then we have

$$\begin{aligned} A_{1,xy} &= A_{1,x} + A_{x,xy} &(1)\\ &= A_{1,x} + A_{1,y} &(2)\end{aligned}$$

where (2) follows by the property proved by Gregorius. Then he made a most interesting remark.

[7] *Solutio problematis a M. Mersenno propositi* (Solution of a Problem Proposed by Mersenne), 1649. This work is not at West Point, but it is bound at the end of the copy of the *Opus geometricum* in the New York Public Library.

He said that this area behaved like a logarithm, for the rule

$$A_{1,xy} = A_{1,x} + A_{1,y}$$

is completely analogous to

$$\ln(xy) = \ln(x) + \ln(y)$$

To us this appears to be only a tiny step, but what looks like a small step to us may not have been small to the creative mathematician who made it. It is very easy to read things into a text, and as historians we must avoid it. What is obvious to us may not be obvious to the person who wrote it.[8]

Our way of introducing the logarithm, i.e., by *defining* it to be

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

was not proposed for use in the schools until the end of the nineteenth-century, when Felix Klein (1849–1925) did it in his *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis*, Dover, p. 156; The German original is from 1908.

This proof of Gregorius is a very interesting way of developing the logarithm. I have used it in class and found it to be quite satisfactory.[9]

[8] We always say that the logarithm is a transcendental function, and this was certainly known to Leibniz and the Bernoulli's, but I do not know who proved it. There is a nice proof by R. W. Hamming, of error correcting code fame, in a paper entitled "An elementary discussion of the transcendetal nature of the elementary transcendental functions," *American Mathematical Monthly*, 77 (1970), pp. 294–297; reprinted in *A Century of Calçulus, Part II 1969–1991*, edited by Tom M. Apostol et al., MAA 1992, pp. 80–83.

[9] Rosemary Schmalz, "A 'natural' approach to e," *The Mathematical Gazette*, vol. 74, #470, December 1990, pp. 370-372 outlines a similar approach without mentioning Gregorius.