History of the Brachistochrone

In April 1696, Isaac Newton, then 53, resigned the Lucasian Professorship at Cambridge to devote the remaining three decades of his life to "ye Kings business" at the Mint in London.[1] Just nine months later, on January 29, 1696/7[2], Newton received a letter from France, probably from Varignon, but possibly L'Hospital, containing two challenge problems. The fly-sheet on which this "Programma" was printed at Groningen was dated 1 January 1697 and addressed

> To the sharpest mathematicians now flourishing throughout the world, greetings from Johann Bernoulli, Professor of Mathematics.

Bernoulli's stated aims in proposing the problem are admirable:

> We are well assured that there is scarcely anything more calculated to rouse noble minds to attempt work conductive to the increase of knowledge than the setting of problems at once difficult and useful, by the solving of which they may attain to personal fame as it were by a specially unique way, and raise for themselves enduring monuments with posterity. For this reason, I ... propose to the most eminent analysts of this age, some problem, by means of which, as though by a touchstone, they might test their own methods, apply their powers, and share with me anything they discovered, in order that each might thereupon receive his due meed of credit when I publically announced the fact. [Scott 1967a, p. 224]

Bernoulli's new year's present to the mathematical world was the problem:

> To determine the curved line joining two given points, situated at different distances from the horizontal and not in the same vertical line, along which a mobile body, running down by its own weight and starting to move from the upper point, will descend most quickly to the lowest point. [Scott 1967a, p. 225]

Johann Bernoulli christened this enigma the brachistochrone problem, a word he coined from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. He added that the solution was not a straight line, but a curve well known to geometers. [At this point I am careful not to reveal the solution to the students.]

The problem was not new. In 1638 Galileo attacked it in his last work *Discourses and Mathematical Demonstrations Concerning Two New Sciences*, generally known in English by the last words of the title. He was only able to prove a circular arc was better than a straight line of descent, although he incorrectly concluded that a circular arc was the solution.

In fact, Bernoulli had stated the problem earlier. How he happened to offer it to the world in 1697 as a New Year's Day present is an interesting part of the story. In the *Acta eruditorum* of June 1696 (pp. 264–269), Bernoulli had attempted to show that the calculus was necessary and sufficient to fill the gaps in classical geometry. At the very end of the paper he tacked on the brachistochrone problem as a challenge. When the six months allotted for the solution were up, Bernoulli had received no correct solutions. He had received a letter from Leibniz praising the problem and indicating that he had solved it in one evening. Indeed he had found the differential equation describing the curve, but he had not yet recognized the curve as an inverted cycloid. Bernoulli and Leibniz interpreted Newton's six month silence to mean the problem had baffled him—indeed he had not seen it. Thus they intended to demonstrate the supe-

[1] I have used this material at the end of a two or three semester calculus sequence. My purpose for presenting it there, rather than in a differential equations course, is that it pulls together many topics from the calculus. It is also a tour de force that presents a beautiful piece of mathematics. On several occasions I have covered the mathematics and history in one fifty minute class period. The historical details given here are somewhat more detailed than I actually present, as they are intended for instructors rather than students. In particular, I try to debunk the myths common in this oft-told tale.

[2] The slash date indicates that the Gregorian calendar was not yet in use in England; the date on the continent was February 8, 1697

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riority of their methods publicly [remember, the priority dispute was just beginning]. Thus Leibniz suggested the deadline be extended to Easter and that it be distributed more widely. So Bernoulli added a second problem[3], had a broadside published, and made sure it circulated widely. This was his New Year's "Programma."

The brachistochrone problem was a difficult one. In France, Pierre Varignon admitted that he was "immediately rebuffed by its difficulty," and L'Hospital pleaded that it would need to be "reduced to pure mathematics" before he could attempt it, "for physics embarrasses me." In Oxford, John Wallis was stumped and David Gregory wasted two months trying to prove that the catenary was the desired curve before Newton set him right.

Thirty years later Newton's niece Catherine Barton Conduitt recalled,

> When the problem in 1697 was sent by Bernoulli— Sr. I. N. was in the midst of the hurry of the great recoinage [and] did not come home till four from the Tower very much tired, but did not sleep till he had solved it wch was by 4 in the morning. [Westfall 1980a, 582–3; Whiteside 1981a, 72–73.]

Although she probably heard this story from Newton later, rather than being a witness herself for she was probably not yet his housekeeper [See Westfall 1980a, p. 595], there is no reason to doubt it. Indeed, the next day Newton sent his answer to his old Cambridge friend Charles Montague, who was then President of the Royal Society. He did not send any justification that the answer was a cycloid [see Scott 1967a, p. 226, or Whiteside 1981a, p. 75, where his "solution" takes but one paragraph]. The original worksheets are not extant [Whiteside 1981a, p. 74], which is surprising given Newton's propensity to save every scrap of paper he ever wrote on. Newton did not instruct Montague on what to do with the answer. Even though the "Programma" explicitly said they should be sent to Bernoulli, Montague immediately had the answer published anonymously in February in the *Transactions of the Royal Society* [vol. 17, no. 224 (for January 1696/7), pp. 384–389]. Thus the trap that Bernoulli and Leibniz had set for Newton failed to snare its game.

Derek T. Whiteside, who has published an extremely valuable edition of *The Mathematical Papers of Isaac Newton* in eight volumes (with an index volume yet to follow), claims that the fact that it took Newton twelve hours to solve these problems indicates that his mathematics was rusty from nine months disuse. It also shows that the gradual decline[4] in Newton's mathematical ability had set in. However, his solution of the brachistochrone problem is one piece of counterevidence to the myth that Newton's old age was mathematically barren [Whiteside, 1981a, pp. xii, 3].

Immediately on receiving the solution of the anonymous Englishman via Basnage de Beauval, Bernoulli wrote Leibniz that he was "firmly confident" that the author was Newton. Leibniz was more cautious, admitting only that the solution was suspiciously Newtonian. Several months later Bernoulli wrote de Beauval that "we know indubitably that the author is the celebrated Mr. Newton; and, besides, it were enough to understand so by this sample, ex ungue Leonem." Within a few weeks this shrewd guess was common knowledge across Europe. Unfortunately the phrase, "from the paw of the lion," was so scrambled (an initial "tamquam" was added) by Newton's biographers Woodhouse and Brewster that it has "travelled from a mere pedestrian *cliché* to be a universally parroted (but no less spurious) myth" [Whiteside 1981a, pp. 9–10]. The phrase goes back to Plutarch and Lucian, who allude to the sculptor Phidias' ability to determine the size of a lion given only its severed paw. In Bernoulli's day the phrase did not carry the power it does today.

Not having succeeded in trapping Newton, Bernoulli quickly lost interest. Thus it was left to Leibniz to publish in the May 1697, *Acta* (pp. 201–224) the solutions received by the deadline. These included Johann's own solution *"Curvatura radii in diaphanis non uniformibus ..."* [The curvature of a ray in nonuniform media], one by his older brother Jakob *"Solutio problematum fraternorum ..."* [Solution of a problem of my brother], one by L'Hospital (probably not completely indepen-

[3] Bernoulli's second New Year's Day problem was: To find a curve such that the sum of the two segments PK and PL, on a line drawn at random from a point P to cut the curve in two points K and L, though the two segments be raised to any power, is a constant.

[4] The brachistochrone problem should not be confused with Bernoulli's challenge problem of December 1715, asking for the family of curves orthogonal to a given family, which was designed to test the "pulse of the English analysts." Newton fared much worse here. See Whiteside 1981a, p. 504.

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dently), one by Tschirnhaus, and a reprint—seven lines in all—of Newton's. This time Newton was not anonymous, for Leibniz had mentioned him in his introductory note (p. 204). Leibniz was so embarrassed by the whole thing that he wrote the Royal Society indicating that he was not the author of the challenge problem. Technically, this was true, but he had contrived with Bernoulli to embarrass Newton.

Within a few years there were solutions by John Craige, David Gregory, and Richard Sault. In 1699 Fatio de Duillier published a solution, though the paper is better remembered for naming Leibniz as "second inventor" of the calculus. By 1704 Charles Hayes, in his widely read *Treatise on Fluxions*, presented it as a mere worked example in a textbook. As often happens, a difficult problem, once cleverly solved, comes within the grasp of many.

Johann's solution[5] was elementary and clever: the path of quickest descent is the same as a light ray passing through a fluid of variable density. This is the solution we will present. It is important because it presents the first example of the optical-mechanical analogy. See Struik 1969a for an English translation and Simmons 1972a for a modern presentation of the mathematics. Jakob's solution was geometrical and laborious. But it was more general and an important early step in the calculus of variations. He realized that this was a new *type* of problem—the variable is a function.

Newton did not think kindly of Bernoulli's challenge for he wrote Flamsteed two years later, "I do not love to be printed upon every occasion much less to be dunned and teezed by forreigners about Mathematical things or to be thought by our own people to be *trifling* away my time about them when I should be about ye Kings business" [Scott 1967a, p. 296]

The Mathematics

History of the Brachistochrone

Imagine a curved wire from A to B and a bead sliding down the wire with friction neglected. Which curve minimizes the descent time? Galileo proved a circular path was better than a straight line path. But that is not the minimum. Here is how Johann Bernoulli solved the brachistochrone problem in 1697.

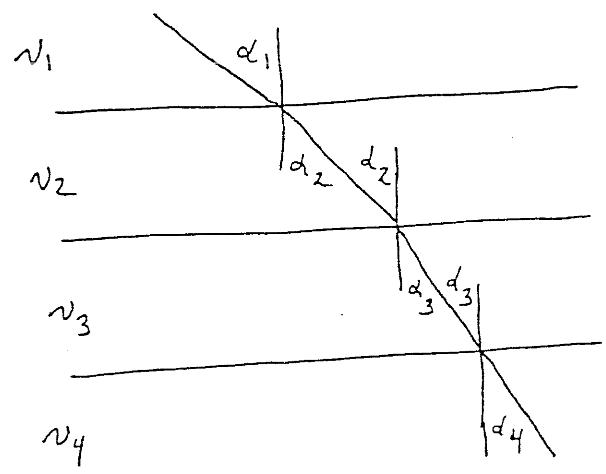
Some Optics: In the differential calculus we considered the behavior of a ray of light as it passed from air into water, and derived Snel's law of optics:

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$

where α_1 is the angle of incidence, α_2 the angle of reflection, and v_1 the velocity of light in air, and v_2 its velocity in water.

Now if we consider what happens to light when it passes through a medium of variable density. Suppose we have a stratified optical medium, with the velocity constant in the individual layers. Then we obtain

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = \frac{\sin\alpha_3}{v_3} = \dots$$



Now if the layers grow thinner and more numerous, we obtain, in the limit,

$$\frac{\sin\alpha}{v} = k \tag{1}$$

[5] Carl Boyer, *A History of Mathematics*, p. 457, indicates that Johannes Bernoulli, "found an incorrect proof that the curve is a cycloid, and he challenged his brother to discover the required curve. After Jacques [Jakob] correctly proved the curve sought is a cycloid, Jean [Johannes] tried to substitute his brother's proof for his own." While this story certainly fits in with the character of the family, I know of no justification for it. Might this relate to the catenary? See Kline, *Mathematical Thought form Ancient to Modern Times*, pp. 472–473.

where k is constant.

Some Mechanics: A freely falling body has constant accelleration, i.e.,

$$\frac{d^2y}{dt^2} = g.$$

Integrating, we obtain

$$v = dy/dt = gt + v_0, \qquad \text{where} \quad v(0) = v_o$$

and

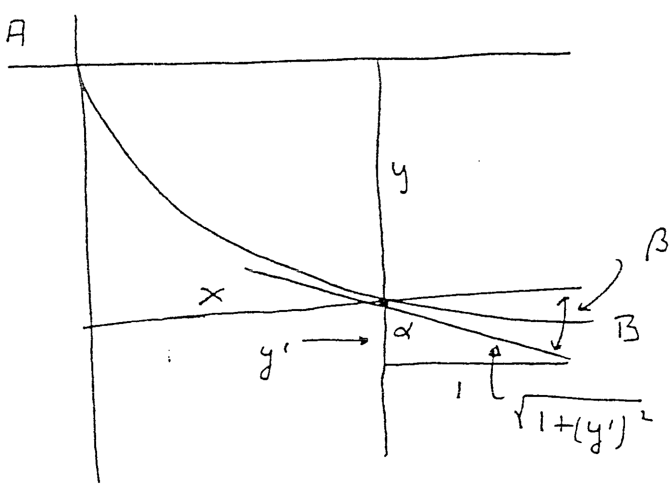
$$y = \frac{1}{2}gt^2 + v_0t + y_0, \qquad \text{where} \quad y(0) = y_o.$$

$v=gt$ If the body falls from rest starting at $y = 0$, then $y = gt$ and $y = \frac{1}{2}gt^2$. Eliminating t, we obtain the useful equation

$$v = \sqrt{2gy}, \tag{2}$$

which gives velocity in terms of the distance fallen.

Some Calculus: From the diagram



one can read off

$$\sin\alpha = \frac{1}{\sqrt{1+(y')^2}} \tag{3}$$

or, if some student insists, it can be derived:

$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}}.$$

Combining (1), (2), and (3) we obtain a differential equation describing the motion of the body sliding down curve:

$$k = \text{constant} = \frac{\sin\alpha}{v} = \frac{1}{\sqrt{2gy}\sqrt{1+(y')^2)}}$$

or

$$y(1+(y')^2) = c, \qquad \text{where} \quad c = 1/2gk^2.$$

To solve this equation, separate the variables and integrate:

$$\int dx = \int \sqrt{\frac{y}{c-y}}\,dy.$$

To evaluate the second integral, use the unusual substitution

$$\tan\theta = \sqrt{\frac{y}{c-y}}$$

which is, after some trigonometry, $y = c\sin^2\theta$. Thus

$$\begin{aligned} x &= \int \sqrt{\frac{y}{c-y}}\,dy \\ &= \int \tan\theta \cdot 2c\sin\theta\,\cos\theta\,d\theta \\ &= c\int 2\sin^2\theta\,d\theta \\ &= c\int (1 - \frac{1}{2}\cos\theta)\,d\theta \\ &= \frac{c}{2}(2\theta - \sin(2\theta)) + c_1. \end{aligned}$$

If we choose our coordinate system so that A is at the origin, then $x(0) = 0$ and $y(0) = 0$. So $\tan\theta = 0$ and $\theta = 0$. Consequently, if $x = y = \theta = 0$, we have $c_1 = 0$. This yields the parametric form of the solution:

$$x = \frac{c}{2}(2\theta - \sin(2\theta))$$

$$y = \frac{c}{2}(1 - \cos(2\theta)),$$

where the last is obtained by a trigonometric identity on the substitution we made. These are the parametric equations of a cycloid where the radius of the generating circle is $c/2$ and θ is ?????.

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but his view of the brachistochrone problem (pp. 105–106) is flawed: He makes Varignon a pupil of Bernoulli and says there is no hint of a desire to score off against Newton.

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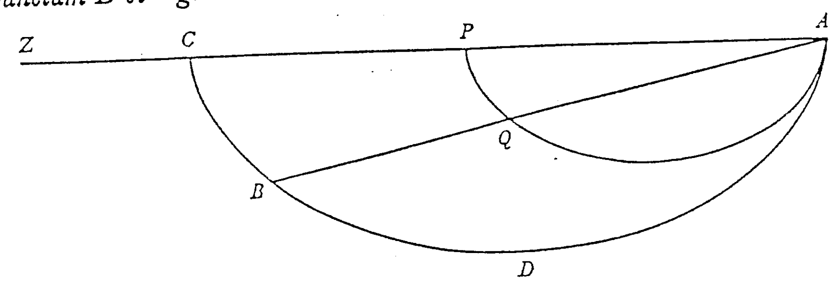
1980a *Never at Rest. A Biography of Isaac Newton*, Cambridge. See pp. 581–583.

Whiteside, D. T.

1981a *The Mathematical Papers of Isaac Newton, Volume VIII, 1697–1722*, Cambridge University Press. Contains excellent and very scholarly commentary by Whiteside (pp. 1–14) as well as the pertinent documents (72–90) and numerous references. This is the primary source of my information.

Biographical information of high quality on all of the individuals mentioned above can be found in the *Dictionary of Scientific Biography*, edited by C. C. Gillispie.

Investiganda est curva Linea ADB in qua grave a dato quovis puncto A ad datum quodvis punctum B vi⁽⁶⁾ *gravitatis suæ citissimè descendet.*



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