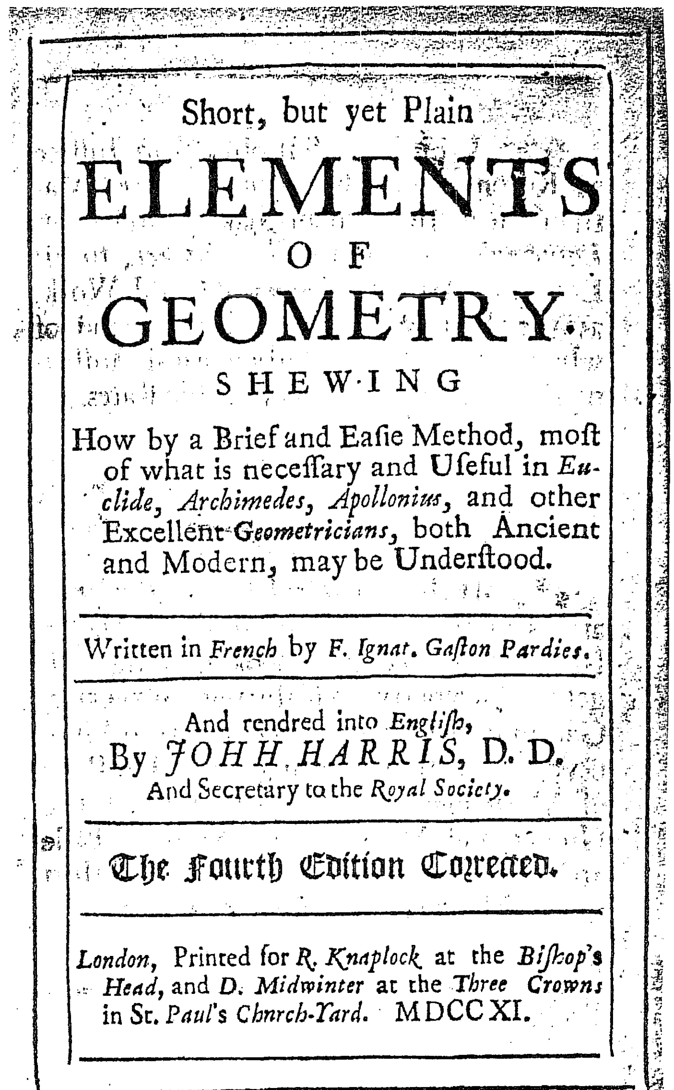
The Bridge and the Catenary

What thing is it,
the less it is the more it is dread?
A riddle, 1511[1]

Ignace Gaston Pardies (1636-1673) was a French Jesuit who corresponded with Huygens and Newton, yet he is little known today among mathematicians, partly because his work was primarily in physics. Yet there are several things he did that deserve our attention. We begin with some comments from his *Élémens de géométrie* (1671) which was translated into Dutch (1690), Latin (1694) and English (1701). The work begins with advice for the reader which is well worth quoting in full (from the 1711 edition), for it is splendid advice for our students today.

> Pardie's Advice to those who would Understant Geometry.
>
> 1. They ought to enure themselves to consider well the *Figures*, at the same time as they Read the *Propositions*. There will be some Labour and Difficulty at first, but they will break thro' it in two or three Days.
> 2. They ought not to be discouraged, if they meet with some things which they do not understand at first; Geometry is not so easie to be attained, as History.
> 3. If, after they have Read and Considered attentively any Proposition, they find they don't understand it; let it be passed over, it will probably be Intelligible by reading further, or at least when they have gone over the whole, and have began to Read it over a-new. There are indeed many things in Geometry, that will never be well understood at first Reading over.
> 4. The Numbers which are within Parenthesis, *v.gr.* (3. 14.) shew that the Matter there spoken of, hath been proved elsewhere, *viz.* in this Instance in the fourteenth Article of the *Third* Book; And they ought always to mind the Number of the Article, and to consult the Places referred to, that so they may gain the Demonstration of what they Read.


Short, but yet Plain
ELEMENTS
OF
GEOMETRY.
SHEWING
How by a Brief and Eaſie Method, moſt of what is neceſſary and Uſeful in *Euclide*, *Archimedes*, *Apollonius*, and other Excellent *Geometricians*, both Ancient and Modern, may be Underſtood.

Written in *French* by *F. Ignat. Gaſton Pardies.*

And rendred into *Engliſh*,
By *JOHH HARRIS*, D. D.
And Secretary to the *Royal Society.*

The Fourth Edition Corrected.

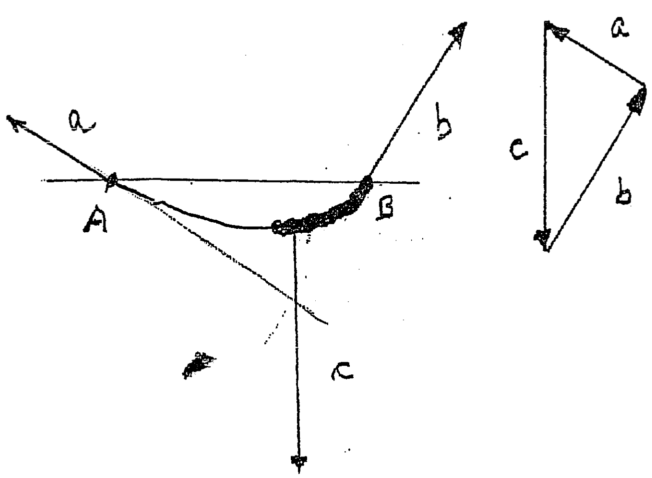
London, Printed for *R. Knaplock* at the *Biſhop's Head*, and *D. Midwinter* at the *Three Crowns* in St. *Paul's Chnrch-Yard.* MDCCXI.

[1] Henry Petroski, *To Engineer is Human* (1985), p. 158. This book has a nice chapter on bridges and the author has several books on the topic. The riddle comes from the first English book on the topic. See Mark Bryant, *Riddles: Ancient and Modern*, New York: Peter Bedrick Books, 1983. The answer to the riddle is a bridge.

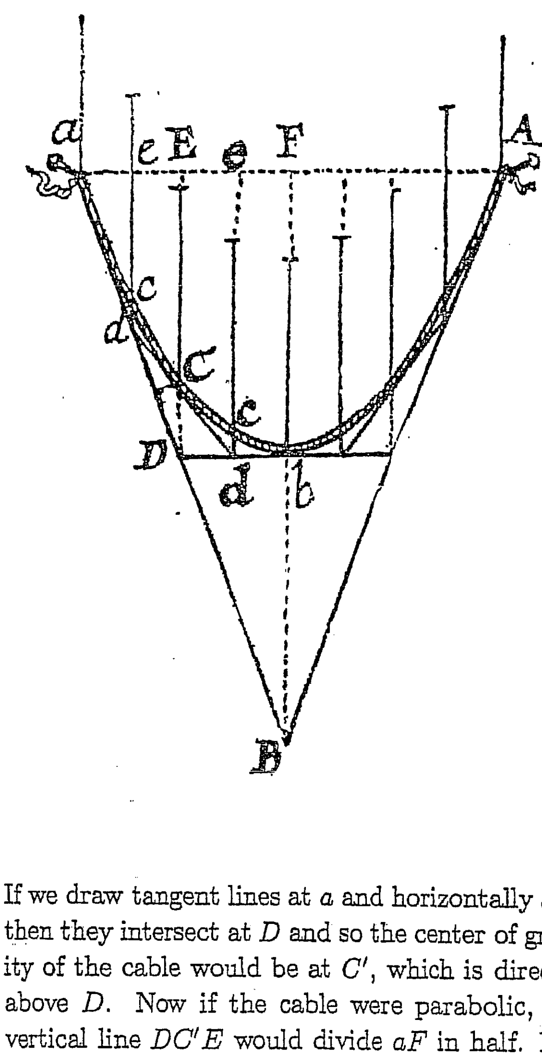
5. When they meet with any Words which they don't understant, they may consult the Table at the End of the Book.

6. 'Tis good to have a Master at first, to Explain to them the Nature and Manner of the Demonstrations: for by that means they will Understand the thing much easier and much sooner, than they can do by Reading by themselves.

Now we turn to the work which is our primary interest, *La Statique ou la Science des Forces Mouvantes*. This, which was first published in 1673, is one of the earliest works on statics and dynamics. One problem set in this book is to find the center of gravity of a hanging cable. If the cable is of uniform density and if the points A and B from which it hangs are at the same horizontal level, then symmetry makes it easy to identify the center of gravity. However, Pardies does not make this assumption, he considers any cable whatsoever, no matter how it is loaded. Although I have asked lots of people, I have not yet found a mathematician, physicist, or engineer who could solve this problem. Yes, Pardies has a remarkable simple solution. For the classroom, we will phrase it using vectors for it is a neat example. Let a and b be the force vectors needed to support the cable at points A and B, which need not be on the same horizontal line. Of necessity, these vectors will be tangent to the cable at its ends. Also draw the vector c, the downward forge acting on the cable due to its weight. This eminates from the center of gravity of the cable. If the cable is to remain stationary, then $a+b+c=0$, i.e., the three vectors form a triangle. Thus to find the center of gravity we extend the force vectors a and b downward from the points A and B from which the cable hangs. The vector c must pass through the intersection of these two lines. This is the simplest real-world application of vectors that I know.



Using this result, Pardies showed that a uniform hanging cable, a catenary, is not a parabola. Consider only the portion of the cable between one end a and the lowest point b.



If we draw tangent lines at a and horizontally at b then they intersect at D and so the center of gravity of the cable would be at C', which is directly above D. Now if the cable were parabolic, the vertical line $DC'E$ would divide aF in half. But the the part aC' of the parabola would be heavier than the part $C'b$, so the center of gravity of the parabola is not at C'. Thus the hanging chain, the catenary, is not a parabola.

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For the seventeenth-century reader this proof is complete, but for the modern reader, who has not read Apollonius,[2] a bit more detail is needed. Consider the parabola $y = ax^2$. If we draw the tangent at the point (x_0, ax_0^2), it will intersect the x-axis at the point $x_0/2$. This is a nice result, for it shows how to actually *draw* the tangent line to a parabola, which almost no one today knows. You should ask your students to get out their calculus and check this result.

After this interlude about the catenary, Pardies considered the following problem: "But if we consider a thread without weight, on which rests an infinity of equally heavy lines EC, ec, parallel and equally distant from each other, then the thread $aCbA$ will be perfectly parabolic."

He does not mention bridges or suspension bridges, but we can see that this is equivalent to that problem. We would prefer to consider the strings to be hung from the cable. Let us consider weightless strings with equal weights attached to them. If we consider these weights to be all at the same height, then what we have is a model of a suspension bridge with the weights representing the roadway.

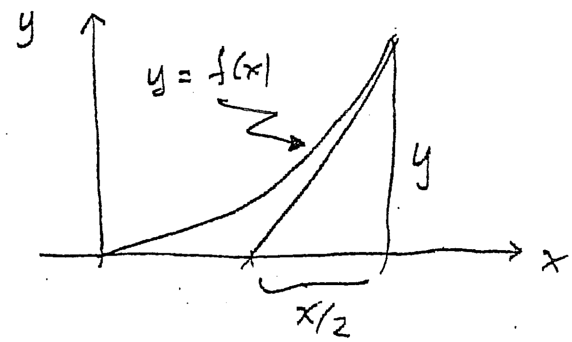
All he does is draw the tangent line aD from end a of the cable, and tangent line bD from the lowest point on the cable. These intersect at D and so the center of gravity must be above it, at C'. Since the vertical lines ec are all equally heavy, the center of gravity must be at C', which is direclty below E and that $aE = EF$. Then he comments "as the geometers know" the curve is a parabola.

Well any seventeenth century geometer who had studied the conics would know this fact, but not many twentieth century mathematicians do, so we shall check it.

Consider the curve $y = f(x)$ in the first quadrant and assume that the tangent line at the point (x, y) will intersect the x-axis at the point $(x/2, 0)$. By definition the slope of this tangent line is $y/(x/2)$ and by calculus it is dy/dx and so we obtain a simple differential equation.

$$\frac{dy}{dx} = \frac{y}{x/2}$$

[2] See Apollonius, Book I, Proposition 33.



Separating variables and integrating we have

$$\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$$

or

$$\ln(y) = 2\ln(x),$$

i.e.,

$$y = x^2.$$

Thus the curve is a parabola.[3]

A Modern Approach

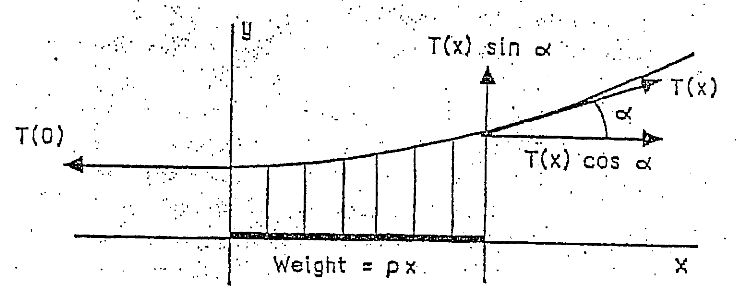
Several years ago, I was teaching an "honors" calculus class, where the emphasis was on concepts, not computation. When we came to the simple differential equations that are discussed in calculus, I asked my students the shape of the cables on a suspension bridge. The answer to this question is not as widely known as it should be. Many mathematics teachers have told me that the shape of the cable on a suspension bridge is a catenary. Let us see why that is wrong.

To build a suspension bridge, erect two tall towers, and construct a cable between them. From this cable we suspend a large number of small vertical cables called suspenders, which are used to support the roadway of the bridge. In an actual bridge, the roadway is almost horizontal, and its

[3] This is the converse of the result of Apollonius cited in note 2. The earliest proof of this result that I know uses calculus and is in Johann Bernoulli's lectures to L'Hospital in 1693, but there must have been earlier geometric proofs.

weight is very large compared to the total weight of the various cables. Thus it is reasonable to ignore the weight of the cables in our mathematical model. This is the crucial idea: only the weight of the roadway matters. It is also important that the weight of the roadway is uniformly distributed in the horizontal direction.

Our goal is to find the shape of the main cable on a suspension bridge. As always, we begin by setting up a convenient coordinate system. Since the cable is obviously symmetric with respect to its low point, let the y-axis pass through this point, and only consider the right portion of the bridge above the interval $[0, x]$. First consider just the cable above this interval. All of the forces acting on this segment of the cable must be in equilibrium or it would be in motion.[4] Let $T(0)$ be the tension on the left end of the cable. Since this is the low point of the cable, the force acts horizontally. Let $T(x)$ denote the tension on the right end of the segment we are considering. This tension vector pulls the cable up and to the right and acts along the tangent line, which is at an angle α with the horizontal. When this vector is resolved into vertical and horizontal components we obtain the situation illustrated.



The assumption that the roadway is uniformly distributed tells us that over the interval $[0, x]$ the roadway has weight ρx, where ρ is the constant density of the bridge deck. This weight acts vertically; it has no horizontal component. Consequently, equating the horizontal forces we obtain the first equation below. The second comes from the vertical forces:

$$T(0) = T(x)\cos(\alpha)$$
$$\rho x = T(x)\sin(\alpha).$$

[4] The most famous counterexample is "Galloping Gertie," the *first* Tacoma Narrows Bridge over Puget Sound in the state of Washington that was torn apart on November 7, 1940 by the wind. Theories abound about what happened ranging from Theodore von Kármán's article in the *Engineering News-Record* just two weeks after it fell to P. J. McKenna, "Large torsional oscillations in suspension bridges revisited: Fixing an old approximation," *American Mathematical Monthly*, 106 (1999) 1–18. Won the MAA's Ford Award in 2000.

What we now desire is a differential equation. All we need to do is remember the basic idea that $dy/dx = \tan\alpha$. Then, from the above equations we obtain:

$$\frac{dy}{dx} = \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\rho x / T(x)}{T(0)/T(x)} = \frac{\rho}{T(0)} x.$$

Integrating we obtain,

$$y = \frac{\rho}{2T(0)} x^2 + h_0.$$

Thus we see that the shape of the cable on a suspension bridge is a parabola.[5]

After I did this problem in class, I said that it was just a warm up for a more difficult and very famous problem, the catenary problem. This time we have only the main cable. What is its shape?

Galileo (1564–1642) had suggested that a heavy rope suspended from both ends would hang in the shape of a parabola, a conjecture which was disproved by Joachim Jungius (1587–1657) and

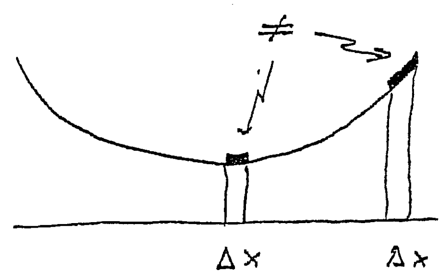
[5] A very nice treatment of this problem along with much general information about bridges is in Alexander J. Hahn, *Basic Calculus From Archimedes to Newton to its Role in Science*, Springer, 1998, pp. 257–265. The book contains a number of other examples of how history can be used in teaching.

published posthumously in 1669. The 1673 disproof of Pardies was given above and Christiaan Huygens (1629–1695) had an unpublished refutation in 1646. If you read the geometric proof of Huygens,[6] you will see what a great accomplishment the new calculus of Leibniz and Newton was. The true shape of the curve was not known until 1690/91 when Huygens, Leibniz, and Johann Bernoulli (1667–1748) replied to a challenge of Jakob Bernoulli (1654–1705). The name "catenary" was introduced by Huygens in a letter to Leibniz in 1690; it derives from the Latin "catena," which means "chain." This was the first independent work of Johann Bernoulli, who was immensely proud that he had solved the catenary problem[7] and that his brother Jakob, who had posed it, had not. Writing to Pierre Remond de Montmort (1678–1719) years later, on 29 September 1718, Johann boasted:

> The efforts of my brother were without success; for my part, I was more fortunate, for I found the skill (I say it without boasting, why should I conceal the truth?) to solve it in full and to reduce it to the rectification of the parabola. It is true that it cost me study that robbed me of rest for an entire night. It was much for those days and for the slight age and practice I then had, but the next morning, filled with joy, I ran to my brother, who was still struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more to try to prove the identity of the catenary with the parabola, since it is entirely false. The parabola indeed serves in the construction of the catenary, but the two curves are so different that one is algebraic, the other is transcendental.[8]

After giving all of this history to my class, time had run out, so I announced that we would omit the derivation of the equation of the catenary. To my great surprise, the students howled in protest and insisted that we do the derivation next time. Naturally, I was happy to oblige, but this event was so unique that I have ever since attributed it to the fact that I had presented the problem in its historical setting. I have no stronger example of history as a motivating force.

So now let us derive the equation of the catenary. The notation in the previous figure, with a few changes, will suffice. Of course, the roadway is no longer there. This time the weight is that of the cable alone. It is distributed uniformly along the cable, not uniformly in the horizontal direction.



This is the main difference from the suspension bridge. The derivation becomes more complicated since we must introduce the parameter s, which denotes arc length. Consequently, the weight of the cable is ρs. The horizontal forces are the same as before, so we have

$$T(0) = T(x)\cos(\alpha),$$

but the downward vertical force is ρs, so this time we have

$$\rho s = T(x)\sin(\alpha).$$

Eliminating $T(x)$ between these two equations and then solving for s we have

$$s = k\tan(\alpha)$$

, where $k = T(0)/\rho$. Thinking ahead, if we differentiate x and y with respect to arc length s we have:

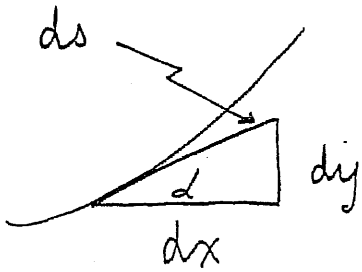
[6] H. J. M. Bos, "Huygens, Christiaan," *Dictionary of Scientific Biography*, volume 6, pp. 597–613, especially p. 601.

[7] *Acta eruditorum*, 1691; 2, vol. 1, 48–51.

[8] *Der Briefwechsel von Johann Bernoulli, (1667–1748)*, edited by Otto Spiess, 1955, pp. 97–98. This translation is from Morris Kline, *Mathematical Thought from Ancient to Modern Times*, 1972, p. 473.

$$\frac{dx}{ds} = \cos(\alpha) \quad \text{and} \quad \frac{dy}{ds} = \sin(\alpha),$$

a result which is clear from Leibniz's characteristic triangle.



By the chain rule (what could be more fitting to use in the catenary problem?),

$$\frac{dx}{d\alpha} = \frac{dx}{ds} \cdot \frac{ds}{d\alpha} = \cos(\alpha) \cdot k\sec^2(\alpha) = k\sec(\alpha)$$
$$\frac{dy}{d\alpha} = \frac{dy}{ds} \cdot \frac{ds}{d\alpha} = \sin(\alpha) \cdot k\sec^2(\alpha) = k\sec(\alpha) \cdot \tan(\alpha).$$

Integrating each of these we have

$$x = k\ln|\sec(\alpha) + \tan(\alpha)| \quad \text{and} \quad y = k\sec(\alpha).$$

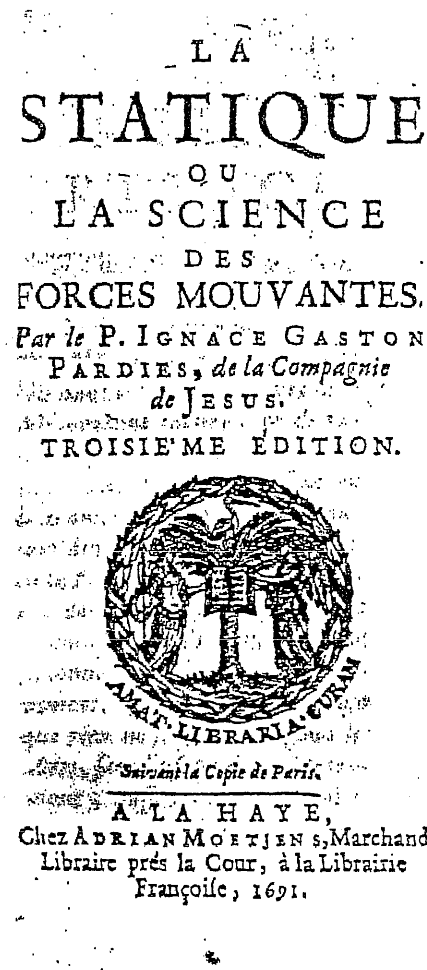
Finally, if we expend a little effort and use some trigonometric identities to eliminate α from these two equations we obtain

$$y = k\cosh(x/k),$$

which is the equation of a catenary.[9]

Thus we have seen that when a suspension bridge is being erected, and only the cable is up, then it assumes the shape of a catenary. However, when the roadway is installed below, then the cable changes shape to a parabola.

Perhaps this is an opportune point to mention the issue of historical accuracy in the classroom. Contrary to the professional historian of mathematics, the classroom teacher need not be a slave to historical details and methods. The teacher should not lie, but it is not necessary to tell the whole story. To provide an overabundance of detail will bore the students and will not advance our goal of using history to motivate and instruct the students. In particular, it is not necessary, and seldom desirable, to use the same methods to derive results that their inventors did. The above derivation for the catenary is stated in modern language, and I would certainly not apologize for doing so in class.[10]


LA
STATIQUE
OU
LA SCIENCE
DES
FORCES MOUVANTES.
Par le P. IGNACE GASTON
PARDIES, de la Compagnie
de JESUS.
TROISIE'ME EDITION.

Suivant la Copie de Paris.
A LA HAYE,
Chez ADRIAN MOETJENS, Marchand
Libraire prés la Cour, à la Librairie
Françoise, 1691.

[9] For a somewhat different solution see C. H. Edwards Jr. and David E. Penney, *Calculus and Analytic Geometry*, 1982, pp. 371–373. Also see Paul Cella, "Reexamining the Catenary." *College Mathematics Journal*, 30 (1999), 391–393.

[10] An earlier version of this note appears in my paper "My favorite ways of using history in teaching calculus," pp. 123–134 in *Learn From the Masters*, edited by Frank Swetz et alia, MAA, 1995.

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