From Kant to Hilbert: A Source Book in the Foundations of Mathematics

Volume II

WILLIAM EWALD
The Law School
University of Pennsylvania

CLARENDON PRESS · OXFORD

19

Georg Cantor (1845–1918)

Ever since Newton mathematicians have grappled with the problem of infinity as it arises in the foundations of the calculus and in the theory of the real numbers. This topic has arisen in many of the selections in Volume 1: recall, for instance, Berkeley's 'Of infinites' (*1707–8*); his critique of infinitesimals and fluxions in *The analyst* (*1734*); MacLaurin's defence of Newton in the *Treatise of fluxions* (*1742*); D'Alembert's comments on infinity, differentials, and limits (*1754*, *1765a*, and *1765b*); and Bolzano's paper on the intermediate value theorem (*1817a*), or his *Paradoxes of the infinite* (*1851*). In the course of the nineteenth century mathematicians like Gauss and Cauchy, Abel and Dirichlet, Fourier and Riemann, Weierstrass and Heine laboured to put real analysis on a rigorous foundation, and by the 1870s they had largely created the modern theory of the elementary calculus, giving precise ε and δ definitions of continuity, differentiability, limits, and the like.

Cantor's investigations grew out of this tradition, and mark a new era in the foundations of mathematics. Cantor, together with Dedekind, created the set-theoretic approach to mathematics that was to dominate the twentieth century; and for sheer unexpectedness, for sheer imaginative power, his theory of transfinite arithmetic has no equal in modern mathematics except possibly the theory of non-Euclidean geometry. The Continuum Hypothesis belongs to a different order of difficulty and depth than Boole's algebraic analysis of the syllogism, or De Morgan's studies of the logic of relations, or Frege and Peirce's discovery of the quantifiers. Those were important events, but in a sense inevitable; whereas the transfinite numbers burst on the world of mathematics entirely unexpectedly. Cantor's work raised profound new questions about the infinite, and was responsible for drawing mainstream mathematical intellects of the calibre of Hilbert, Poincaré, Hausdorff, Brouwer, Weyl, and von Neumann to the foundations of mathematics.

The following selections represent but a small part of Cantor's set-theoretical *œuvre*, and have been selected for the light they shed on the sources and development of his frequently murky thought. Those seeking a detailed examination of Cantor's researches and his struggles with the Continuum Hypothesis should consult the study by Michael Hallett, *Cantorian set theory and limitation of size* (*Hallett 1984*).

Cantor was born in St Petersburg, and schooled in Wiesbaden and Darmstadt. He entered the University of Berlin in 1863, where he received his mathematical education. His teachers at Berlin were Kummer, Kronecker, and above all Karl Weierstrass, whose lectures on the foundations of real analysis exerted a strong

influence on Cantor's early papers. Cantor left Berlin in 1869 to assume a teaching position at the University of Halle; he remained in Halle for the rest of his life.

Cantor's early mathematical papers investigated Fourier series and the foundations of real analysis; it was this research that eventually led him to his research in the theory of sets. He was influenced in this early work by Riemann's 1854 *Habilitation* thesis, 'On the representation of a function by a trigonometric series', which had been posthumously published by Dedekind in 1868. (This paper, in addition to examining the convergence properties of Fourier series, also treated problems in infinitesimal analysis and stated the definition of the Riemann integral.) Cantor investigated the unique representability of functions by trigonometric series, proving in *Cantor 1870* that, if $f(x)$ is represented by a trigonometric series convergent for all x, then the representation is unique. In *Cantor 1871* he strengthened the result, showing that his theorem still holds even if the series diverges at a *finite* number of points in any given interval.

His next important paper, *Cantor 1872*, extended these results yet further; it made major contributions to classical analysis, and laid the groundwork for his study of infinite sets. Cantor begins in §1 of this paper by sketching his theory of the real numbers: a real number is an infinite series of rational numbers $a_1, a_2, \ldots, a_n, \ldots$ such that for any given ε there exists an n_1 such that, for $n \geq n_1$ and for any positive integer m, $|a_{n+m} - a_n| < \varepsilon$. (This theory is discussed by Cantor, and compared with the theories of Heine and Dedekind, in §9 of *Cantor 1883d*, translated below.) In §2, Cantor defines the notion of a 'boundary point' of a point-set P to be any point such that every neighbourhood of the point contains infinitely many points of P. The *first derivative* of P (designated by P') is the set of all boundary points of P; the second derivative P'' is the set of all boundary points of P', and so on. In §3 Cantor extends his earlier results on trigonometric series: uniqueness of representation holds even if the series diverges at an infinite number of points, provided the set of points is of finite order. (A point-set P is of finite order if, for some integer n, the nth derivative $P^{(n)}$ of P is a finite set.) The paper of 1872 laid the foundations of point-set topology (its techniques were adopted in particular by Haudsorff, Borel, and Fréchet); it also led Cantor into a deeper study of the cardinality of subsets of the real line and (in his *1880*) to the study of *infinite* derivatives of a point-set.

A. ON A PROPERTY OF THE SET OF REAL ALGEBRAIC NUMBERS (*CANTOR 1874*)

This article is Cantor's first published contribution to the theory of sets. The deep and epoch-making result of the paper is not the easy theorem alluded to in the title—the theorem that that the class of real algebraic numbers is countable—but rather the proof, in §2, that the class of real numbers is *not* countable. It was this result that first gave a clear sense to the idea that infinite sets could be of different sizes, and that marks the start of the theory of the transfinite. The background to Cantor's discovery can be found in the next selection below, in the opening letters of his correspondence with Dedekind.

It should be noted that Cantor's proof here is *not* the familiar 'diagonal argument', an argument he did not publish until 1891. (*Cantor 1891* is translated below.) Cantor's proof in this paper is rather based on the Bolzano–Weierstrass theorem, of which he had given a proof in *Cantor 1872*; unlike the diagonal argument, it does not extend to sets in general.

The translation is by William Ewald; references to *Cantor 1874* should be to the section numbers, which appeared in the original text.

By a real algebraic number one generally understands a real number ω which satisfies a non-constant equation of the form:

$$a_0\omega^n + a_1\omega^{n-1} + \ldots + a_n = 0, \tag{1}$$

where $n, a_0, a_1, \ldots a_n$ are integers: we can here imagine that the numbers n and a_0 are positive, the coefficients $a_0, a_1, \ldots a_n$ are without common parts, and the equation (1) is irreducible; with these stipulations it turns out that, by the well-known fundamental theorems of arithmetic and algebra, equation (1), which is satisfied by a real algebraic number, is fully determinate; conversely, if n is the degree of an equation of the form (1), then the equation is satisfied by at most n real algebraic numbers ω. The real algebraic numbers form in their totality a set ⟦Inbegriff⟧ of numbers, which shall be designated by (ω); as is readily seen, (ω) has the property that in every neighbourhood of any given number α there are infinitely many numbers from (ω). So it ought at first glance to be all the more striking that one can correlate the set (ω) one-to-one with the set (designated by the sign (ν)) of all positive integers ν—in such a way that to every algebraic number ω there corresponds a definite positive integer, and, conversely, to every positive integer ν there corresponds an entirely definite real algebraic number ω. Or, to say the same thing in different words, the set (ω) can be thought of in the form of an infinite lawlike sequence ⟦gesetzmäßigen Reihe⟧

$$\omega_1, \omega_2 \ldots \omega_\nu, \ldots \tag{2}$$

in which all the individuals of (ω) appear and each of which occurs in (2) at a definite position, which is given by the accompanying index. As soon as one has found a law by which such a correlation can be thought, it can be modified at will; so in §1 I shall describe the correlation which seems to me the least complicated.

In order to give an application of this property of the set of all real algebraic numbers I show in §2 that, given any arbitrarily chosen sequence of real numbers of the form (2), then, in any given interval $(\alpha \ldots \beta)$, one can determine ⟦bestimmen⟧ numbers η which are *not* contained in (2); if one combines the results of these last two paragraphs, then one has a new proof of the theorem, first proved by Liouville, that in any given interval $(\alpha \ldots \beta)$ there are infinitely many *transcendental* (that is, not algebraic) real numbers. Further, the theorem in §2 turns out to be the reason why sets of real numbers which form a so-called continuum (say, all real numbers which are ≥ 0 and ≤ 1) cannot be mapped one-to-one onto the set (ν); thus I have discovered the difference between a so-called continuum and any set like the totality of real algebraic numbers.

§1.

Let us return to equation (1), which is satisfied by an algebraic number ω and which, under the mentioned stipulations, is fully determinate. Then the sum of the absolute values of its coefficients, plus the number $n - 1$ (where n is the degree of ω) shall be called the *height* of the number ω. Let it be designated by N. That is, in the usual notation:

$$N = n - 1 + |a_0| + |a_1| + \ldots + |a_n|. \tag{3}$$

The height N is thus for every real algebraic number ω a definite positive integer; conversely, for every positive integral value of N there are only finitely many algebraic real numbers with height N; let this number be $\phi(N)$; then, for instance, $\phi(1) = 1$; $\phi(2) = 2$; $\phi(3) = 4$. The numbers of the set (ω), i.e. all the algebraic real numbers, can then be ordered in the following manner: one takes as the first number ω_1 the one number with the height $N = 1$; let the $\phi(2) = 2$ algebraic real numbers with the height $N = 2$ follow them according to size, and designate them by ω_2, ω_3; these are followed by the $\phi(3) = 4$ numbers with height $N=3$, according to size; in general, after all the numbers in (ω) up to a certain height $N = N_1$ have been enumerated ⟦abgezählt⟧ and assigned to a definite place, the real algebraic numbers with the height $N = N_1 + 1$ follow them according to size; thus one obtains the set (ω) of all real algebraic numbers in the form:

$$\omega_1, \omega_2, \ldots \omega_\nu, \ldots$$

One can, with respect to this ordering ⟦Anordnung⟧, speak of the νth real algebraic number; not a single member of the set has been omitted.

§2.

Suppose we have an infinite sequence of real numbers,

$$\omega_1, \omega_2, \ldots \omega_\nu, \ldots \tag{4}$$

where the sequence is given according to any law and where the numbers are distinct from each other. Then in any given interval $(\alpha \ldots \beta)$ a number η (and consequently infinitely many such numbers) can be determined such that it does not occur in the series (4); this shall now be proved.

We go to the end of the interval ($\alpha \ldots \beta$), which has been given to us arbitrarily and in which $\alpha < \beta$; the first two numbers of our sequence (4) which lie in the interior of this interval (with the exception of the boundaries), can be designated by α', β', letting $\alpha' < \beta'$; similarly let us designate the first two numbers of our sequence which lie in the interior of ($\alpha' \ldots \beta'$) by α'', β'', and let $\alpha'' < \beta''$; and in the same way one constructs the next interval ($\alpha''' \ldots \beta'''$), and so on. Here therefore α', $\alpha'' \ldots$ are by definition determinate numbers of our sequence (4), whose indices are continually increasing; the same goes for the sequence β', $\beta'' \ldots$; furthermore, the numbers α', α'', ... are always increasing in size, while the numbers β', β'', ... are always decreasing in size. Of the intervals ($\alpha \ldots \beta$), ($\alpha' \ldots \beta'$), ($\alpha'' \ldots \beta''$), ... each encloses all of those that follow.—Now here only two cases are conceivable.

Either the number of intervals so formed is finite; in which case, let the last of them be ($\alpha^{(\nu)} \ldots \beta^{(\nu)}$). Since in its interior there can be at most one number of the sequence (4), a number η can be chosen from this interval which is not contained in (4), thereby proving the theorem for this case.—

Or the number of constructed intervals is infinite. Then the numbers α, α', α'', ..., because they are always increasing in size without growing into the infinite, have a determinate boundary value α^∞; the same holds for the numbers β, β', β'', ... because they are always decreasing in size. Let their boundary value be β^∞. If $\alpha^\infty = \beta^\infty$ (a case that constantly occurs with the set (ω) of all real algebraic numbers), then one easily persuades oneself, if one only looks back to the definition of the intervals, that the number $\eta = \alpha^\infty = \beta^\infty$ *cannot* be contained in our sequence;[1] but if $\alpha^\infty < \beta^\infty$ then every number η in the interior of the interval ($\alpha^\infty \ldots \beta^\infty$) or also on its boundaries satisfies the requirement that it not be contained in the sequence (4).—

The theorems proved in this article admit of extensions in various directions, of which I shall mention only one here:

'If $\omega_1, \omega_2, \ldots \omega_n, \ldots$ is a finite or infinite sequence of numbers which are linearly independent from one another (so that no equation of the form $a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n = 0$ is possible with integral coefficients which do not all vanish) and if one imagines the set (Ω) of all those numbers Ω which can be represented as rational functions with integral coefficients of the given numbers ω, then in every interval ($\alpha \ldots \beta$) there are infinitely many numbers which are not contained in (Ω).'

In fact one persuades oneself through a method of proof similar to that in §1 that the set (Ω) can be conceived in the sequential form

$$\Omega_1, \Omega_2, \ldots \Omega_\nu, \ldots$$

from which, in view of §2, the correctness of the theorem follows.

[1] If the number η were contained in our sequence, then one would have $\eta = \omega_p$, where p is a definite index. But this is not possible, for ω_p does *not* lie in the interior of the interval ($\alpha^{(p)} \ldots$

A quite special case of the theorem cited here (in which the sequence ω_1, $\omega_2, \ldots \omega_n, \ldots$ is finite and the degree of the rational functions, which yield the set (Ω), is stipulated in advance) has been proved, with recourse to Galoisian principles, by Herr B. Minnigerode (See *Math. Annalen*, Vol. 4, p. 497).

B. THE EARLY CORRESPONDENCE BETWEEN CANTOR AND DEDEKIND

Cantor and Dedekind exchanged letters over a period of many years. The mathematical portions of some of their letters were published by Emmy Noether and Jean Cavaillès in 1937; the mathematical portions of others were published (with many errors of transcription) by Ernst Zermelo in his edition of Cantor's writings (*Cantor 1932*). The Cantor *Nachlass*, containing the original letters from Dedekind, appears to have been destroyed in the Second World War. The letters from Cantor and drafts of some of the letters from Dedekind were taken to America by Emmy Noether, and are now kept at the library of the University of Evansville, Evansville, Indiana. (For an account of the history and the contents of the Evansville collection, see *Grattan-Guinness 1974*.) Portions of the non-mathematical correspondence, not published by Noether and Cavaillès or by Zermelo, are reproduced in *Grattan-Guinness 1974* and in the appendices to *Dugac 1976*.

The selection that follows is a translation of the correspondence published by Noether and Cavaillès, omitting only Cantor's letter to Dedekind of 22 December 1879. That letter gives a counterexample to a theorem of Appell in trigonometric series, and is of little importance today. (Cantor's late correspondence with Dedekind in 1899 is translated below as Selection E.) The correspondence spans the period from April 1872 (immediately after the publication of the theories of Dedekind and Cantor on the irrational numbers) to November 1882 (immediately before the publication of Cantor's *Foundations of a general theory of manifolds*). This was the period during which Cantor made his greatest discoveries in the theory of sets, and his correspondence with Dedekind provides rare documentation of the way in which a new mathematical theory was discovered.

The translation is by William Ewald; references should be to the dates of the correspondence.

Cantor to Dedekind

Halle a/S, 28 April 1872

I thank you most warmly for kindly sending me your treatise on continuity and