Some History of the Calculus of the Trigonometric Functions

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1 Abstract

In 1735, Euler found the differential equation $k^4 \frac{d^4y}{d^4x} = y$ to be "rather slippery." In 1739, he "rather unexpectedly" found the full solution, a solution involving trigonometric *functions*. Previously there were only trigonometric "lines" in a circle. Euler's views on trigonometry matured and in 1748 in his *Introductio in analysin infinitorum*, he introduced the trigonometic functions on the unit circle just the way we introduced them today. By the time he published his *Institutionum calculi integralis* (three volumes, 1768-1770), he had a full command of the solutions of first-order linear differential equations with constant coefficients.

2 Abstract

Can you evaluate the integral of the sine using Riemann sums? Do you think Archimedes could? Is it intuitively clear to you that the derivative of the sine is the cosine? If not, why not? What did Newton and Leibniz know about sines and cosines? When did sines become the sine function? Who is the most important individual in the history of trigonometry? Answers will be provided.

3 Archimedes

No doubt you consider Archimedes to be the greatest mathematician of antiquity and one of the greatest of all time, alongside Newton, Euler and Gauss. Probably this belief is based on what you have read and been told. I would like to begin by convincing you that it is true, even though this seems to have little to do with my theme of the history of the calculus of the trigonometric functions.

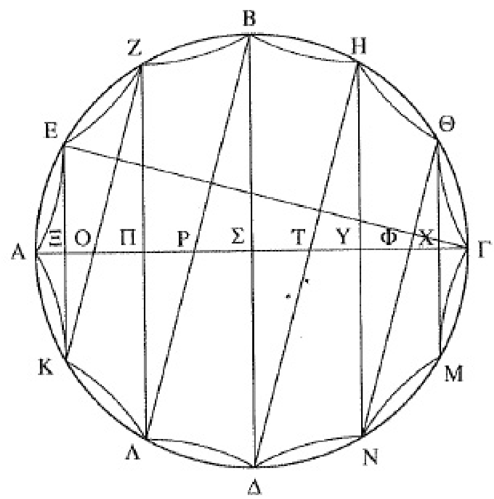
In his work *On the Sphere and the Cylinder, I*, Archimedes has the following proposition:

Proposition 21. If in an even-sided and equilateral polygon is inscribed inside a circle, and the lines are draw through, joining the sides of the polygon (so that they are parallel to one — whichever — of the lines subtended by two sides of the polygon), all the joined <lines> have to the same diameter of the circle that ratio, which the <line> (subtending the <sides, whose number is> smaller by one, than half <the sides>) <has> to the side of the polygon.

Proof: Let there be a circle, $AB\Gamma\Delta$, and let a polygon be inscribed in it, $AEZBH\Theta\Gamma MN\Delta\Lambda K$, and let EK, $Z\Lambda$, $B\Delta$, HN, ΘM be joined; (1) so it is clear that they are parallel to the <line> subtended by two sides of the polygon; Now I say that all the said <lines> have to the diameter of the circle, $A\Lambda$, the same ratio as ΓE to EA.

(a) For let ZK, ΛB, $H\Delta$, ΘN be joined; (2) therefore ZK is parallel to EA (3) while $B\Lambda$ <is parallel> to ZK (4) and yet again to ΔH to $B\Lambda$, (5) and ΘN to ΔH (6) while ΓM <is parallel> to ΘN [(7) and since EA, KZ are two parallels, (8) and EK, AO are two lines drawn through]; (9) therefore it is: as $E\Xi$ to ΞA, $K\Xi$ to ΞO. (10) But as $K\Xi$ to $\Xi 0$, $Z\Pi$ to ΠO, (11) and as $Z\Pi$ to ΠO, $\Lambda\Pi$ to ΠP, (12) and as $\Lambda\Pi$ to ΠP, so $B\Sigma$ to ΣP, (13) and yet again, as $B\Sigma$ to ΣP, $\Delta\Sigma$ to ΣT, (14) while as $\Delta\Sigma$ to ΣT, HY to YT, (15) and yet again, as HY to YT, NY to $Y\Phi$, (16) while as NY to $Y\Phi$, ΘX to $X\Phi$, (17) and yet again, as ΘX to $X\Phi$, MX to $X\Gamma$ [(18) and therefore all are to all, as one of the ratios to one]; (19) and therefore $E\Xi$ to ΞA, so EK, $Z\Lambda$, $B\Delta$, HN, ΘM to the diameter $A\Gamma$. (20) But as $E\Xi$ to ΞA, so ΓE to EA; (21) therefore it will be also: as ΓE to EA, so all the <lines> EK, $Z\Gamma$, $B\Delta$, NH, ΘM to the diameter $A\Lambda$.[1]

[1] [10, p. 112–113]. Reviel Netz, who is editing a new edition of Archimedes, says that this proposition is "strange." The diagram with the sides of the polygon drawn with curved lines tells us something about the way Archimedes drew diagrams. The long name for the dodecahedron is "playful." The proof is a list of facts, it does not "argue." Moreover, it has nothing to do with spheres or cylinders. The many words interpolated in angle brackets are explained thus: "Greek mathematical proofs always refer to concrete objects, realized in the diagram. Because Greek has a definite article with a rich morphology, it can elide the reference to the objects, leaving the definite article alone." (p. 6).



This proof looks horrible, yet it is really quite simple, especially if we rewrite it in more modern notation. Observe that in the diagram, the diameter is divided into 10 line segments, and on each there is a triangle, half below the diameter, half above. All of these triangles are similar and steps 9–17 of the proof use this to obtain:

$$\frac{E\Xi}{\Xi A} = \frac{K\Xi}{\Xi O} = \frac{Z\Pi}{\Pi O} = \frac{\Lambda\Pi}{\Pi P} = \frac{B\Sigma}{\Sigma P} = \frac{\Delta\Sigma}{\Sigma T} = \frac{HY}{YT} = \frac{NY}{Y\Phi} = \frac{\Theta X}{X\Phi} = \frac{MX}{X\Gamma}$$

Now by the theory of proportion — get out your algebra and check — we have

$$\frac{E\Xi}{\Xi A} = \frac{EK + Z\Lambda + B\Delta + HN + \Theta M}{A\Gamma}$$

and by step 20 we have $E\Xi : \Xi A :: \Gamma E : EA$ and thus we reach the final conclusion:

$$\frac{EK + Z\Lambda + B\Delta + HN + \Theta M}{A\Gamma} = \frac{\Gamma E}{EA}.$$

This helps, but it sure does not yet look like calculus. Let us now generalize and rewrite this result in terms of trigonometry. Suppose we have a polygon with $2 \cdot n$ sides rather than the $2 \cdot 6$ that Archimedes has. Thus $\angle E\Gamma A = \pi/n$. Also suppose the radius of the circle is 1. Then we have $EK = 2\sin\pi/n$, $Z\Lambda = 2\sin 2\pi/n$, $B\Delta = 2\sin 3\pi/n$, etc. We also have $\Lambda E = 2\cos\pi/n$ and $EA = 2\sin\pi/2$. Thus our proposition becomes

$$\frac{1}{2}\Big(2\sin\frac{\pi}{n} + 2\sin\frac{2\pi}{n} + 2\sin\frac{3\pi}{n} + \cdots + 2\sin\frac{(n-1)\pi}{n}\Big) = \cot\frac{\pi}{n}$$

Now this looks a lot like a Riemann sum, and if we add one more term and multiply by π/n then we actually have one:

$$\frac{\pi}{n}\Big(2\sin\frac{\pi}{n}+2\sin\frac{2\pi}{n}+2\sin\frac{3\pi}{n}+\cdots+2\sin\frac{(n-1)\pi}{n}+2\sin\frac{n\pi}{n}\Big) = \frac{\pi}{n}\cot\frac{\pi}{n}+\frac{\pi}{n}2\sin\frac{n\pi}{n}$$

Now, taking the limit as n tends to infinity, we have

$$\int_0^\pi \sin x \, dx = 2.$$

If Proposition 22 is rewritten in a similar fashion, we obtain

$$\int_0^\alpha \sin x \, dx = 1 - \cos \alpha.$$

That is rather amazing! Riemann sums are used to evaluate the integral of the sin; you will not find this in many textbooks.[2] Now you will note that I have not claimed that Archimedes knew the integral of the sine. Sir Thomas Little Heath, whose edition of Archimedes is most accessible (since Dover has reprinted it), claims that "Archimedes' procedure is the equivalent of a genuine integration" [5, p. cxlvi]. Well, perhaps so, but 'equivalent' is a very strange yet strong word to be used this way. It is easy for us to read thoughts and interpretations into a text that would be very foreign to the author.

In particular, no claim has ever been made that Archimedes knew about trigonometry. Hipparchus (150 BC), who flourished some sixty years after the death of Archimedes, was the father of trigonometry. His work "On the theory of the lines in the circle," contained a table of chords (double-sines) which he used to compute the rising and setting times of fixed stars. Thus trigonometry had its roots in astronomy.[3] Three centuries later, Ptolemy provided a well developed presentation of trigonometry in his *Almagest* (150 CE).

So who rewrote Archimedes in terms of trigonometry. Heath [5, p. cxlv] cites a work of Gina Loria, but it gives no reference at all and certainly does not appear to be an original contribution of Loria.[4]

NB: Combine Cardano, Roberval etc into a short paragraph above.

4 Cardano

CARDANO, Girolamo (1501-76) De subtilitate libri XXI... - Basel, Ludovicus Lucius, 1554. Early edition of Cardano's most celebrated work (1st edn.: 1550).

[2] It can be found in Richard Courant, *Differential and Integral Calculus* (second edition, 1937, pp. 86–87), Al Shenk, *Calculus and Analytic Geometry*, second edition, 1979, p. 228, and I. M. Gelfand and Mark Saul, *Trigonometry* (Birkhäuser, pp. 169, 228–229). But none of them connect this result with Archimedes. The easiest way to obtain this result is to take the imaginary part of the geometric series $e^{ix} + e^{2ix} + \cdots + e^{nix}$ (Courant, p. 436)

[3] In fact, it was not until late in the sixteenth century that trigonometry was used for land measurement [8]. This was in the *Trigonometria: sive de solutione triangulorum tractatus brevis et perspicuus* (1595), of Bartholomew Pitiscus (1561–1613), the work which coined the word trigonometry. The second edition of this work, *Trigonometriae sive de dimensione triangulorum libri quinque*, contained tables of all six trigonometric functions (the only earlier work to do this was that of Rheticus, 1596).

[4] Heath takes his formulas from Loria, *Il periodo aureo della geometria greca*, p. 108. I have not been able to examine this work. It appears to be the subtitle to volume 2 of *Le scienze esatte nell' antica Grecia* (3 volumes, 1895). However, the relevant formulas appear on pp. 296–297 of the second edition (1914), which is available in the University of Michigan Historical Math Collection: `http://name.umdl.umich.edu/ACU8840.0001.001`.

(This) encyclopaedia... is a mine of facts, both real and imaginary of notes on the state of the sciences; of superstition, technology, alchemy and various branches of the occult (DSB III, 66).

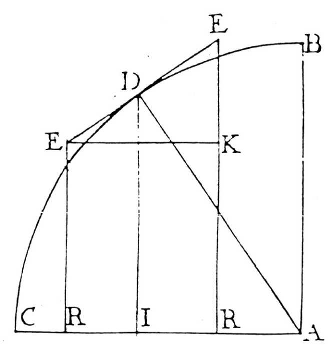
5 Pascal

In June of 1658, Blaise Pascal (1623–1662) had a toothache. To distract himself from the pain he began to think about certain problems of length, volume, and center of gravity connected with the cycloid. When the pain quickly subsided, he took this as a sign from God that the study of mathematics was acceptable and focused on mathematics for the next nine months.

Pascal was convinced that he had refined and broadened Roberval's method of indivisibles and so used his results to challenge the abilities of his contemporary mathematicians in an unsigned circular of June 1658. Due to the short time interval allotted for solutions, no one solved the problems to Pascal's satisfaction. Consequently, Pascal published the solutions in four letters which were collected and republished as the *Lettres de A. Dettonville contenant quelques-unes de ses inventions de geometrie*.[5]

Of special interest in the classroom is a result contained in a portion of the *Lettres* entitled *Traité des sinus du quart de cercle*,[6] which does not deal with the cycloid. In Proposition I, Pascal finds the integral of the sine in a most ingenious way. Pascal begins with a diagram and immediately comments:

> I say that the rectangle formed by the sine DI and the tangent EE is equal to the rectangle formed by a portion of the base (enclosed between the parallels) and the radius AB.

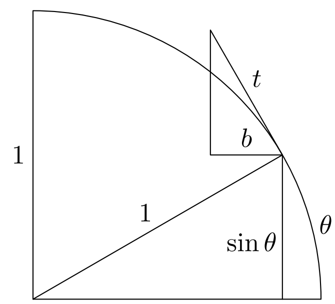


What Pascal then did was to divide the circular arc into infinitely many pieces of equal size and then sum the rectangles. For the modern interpreter it is perhaps better if we do not do things precisely Pascal's way, but rather flip the diagram over and then split the regions he is using into two.

[5]This second order pseudonym, 'Amos Dettonville,' is an anagram, assuming $u = v$, for 'Louis de Montalte,' the pseudonym that Pascal used for his famous *Lettres provinciales*.

[6]*Lettres de A. Dettonville, 1659*, London: Dawsons of Pall Mall, 1966

Consider the following diagram where t is the length of the line segment which is tangent to the circle, b is the base of the indicated triangle, and θ is the length of the arc from the x-axis to the base of the triangle.



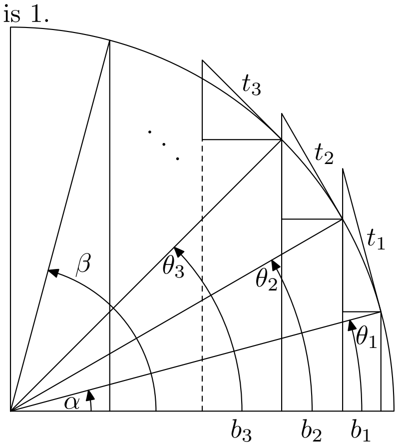
by similar triangles we have

$$\frac{t}{b} = \frac{1}{\sin\theta}$$

so

$$\sin\theta \cdot t = b.$$

Note that this is a restatement of Pascal's remark quoted above in the special case where the radius is 1.



Now consider the sector of the circle bounded by α and β. Divide this arc into infinitely many infinitesimal pieces of width $\Delta\theta$ (only three of which are shown) and draw the indicated tangent lines t_i which begin at the point θ_i. Summing these we have

$$\sum \sin\theta_i \cdot t = \sum b_i.$$

Using the approximation that $\Delta\theta$ is close to t_i and taking the limit as $\Delta\theta$ tends to 0 we have:

$$\lim \sum \theta_i \cdot \Delta\theta = \lim \sum b_i.$$

On the left we have a Riemann sum, and on the right — if we remember that line segments from the origin out to some point on the x-axis are cosines, we see that

$$\int_{\alpha}^{\beta} \sin\theta \, d\theta = \cos\alpha - \cos\beta.$$

Although the method used here is somewhat foreign to modern mathematicians, it does appeal to our students and provides a remarkably easy proof (and, yes, it can be made into a rigorous ϵ-δ proof if you insist on obscuring it).

When the modern mathematician considers this proof, the first puzzling thing is that there is no sine curve in sight. The explanation for this is simple: there was no sine curve at the time. In Pascal's time, a sine was still a certain line segment inside a circle.

6 Newton

In the fall of 1665, Isaac Newton (1642–1727) made annotations on his reading of the *Arithmetica infinitorum* of John Wallis. Using interpolation techniques of Wallis, Newton comes very close to getting the power series for the arcsine. He is reasonably clear about having it, but he does not write it down [11, I, 123–124]. However, when he wrote *De analysi* in 1669 he explicitly gives the series for sine and cosine [11, II, 237], but he does not explain how he obtained them. Although these series were known earlier in India, there is no question that Newton's discovery was original.

[12, II, 35–36]

$$\arcsin x = x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} + \text{etc.}$$

$$\sin z = z - \frac{z^3}{6r^3} + \frac{x^5}{120r^4} - \frac{x^7}{5040r^6} + \frac{x^9}{368220r^8} - \text{etc.}$$

These formulas look strange to you; what is that 'r' doing in there? That is the radius of his circle.

The "circle area sought will be"

$$ax + \frac{x^3}{2a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} \cdots.$$

In 1663 James Gregory (1638–1675), at age 24, traveled to London to oversee the publication of his first book, *Optica promota*, which was a masterly account of mirrors and lenses, containing the first description of a reflecting telescope. While there he met John Collins and they became lifelong friends. Collins was an avid correspondent with scientists and was dubbed "Mersennus Anglus" by Isaac Barrow. He also met Georg Mohr.

Gregory "could have learned in Italy" that the area under the curve $y = 1/(1+x^2)$ is an arctangent. By long division, one obtains $1 - x^2 + x^4 - x^6 + \cdots$, and then by Cavaliari's formula for integrating powers, he easily obtains

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

7 Cotes

In an 1865 text on the differential calculus, John Spare has a section on "Differentiation of the circular functions." He begins by noting that he "presupposes the principle of Analytic Trigonometry" [13, p. 206]. He then finds the derivative of the sine function in a way that closely parallels Cotes:

> 224. In order to differentiate $\sin .x$, we must have for radius 1, if a be any arc, and b any additional arc, by the ratio of corresponding parts of right angled plane triangles:
>
> $$\sin .(a+b) - \sin .a : \tan .b :: \cos .a : 1;$$
>
> that is,
>
> $$\sin .(x+h) - \sin .x : \tan .h :: \cos .x : 1;$$
>
> but if $h = 0$, $\sin .(x+h) - \sin .x$ becomes $d \sin .x$, and $\tan .x$ becomes dx, or differential of the arc.
>
> $$\therefore d \sin .x : dx :: \cos .x : 1$$
>
> $$\therefore d \sin .x = \cos .x \, dx.$$

8 Euler

Euler's work where he creates trigonometric FUNCTIONS because he needs them to solve differential equations.

On 15 September 1739, Euler, in a letter to Johann Bernoulli, reports that he has begun the general treatment of homogeneous linear differential equations with constant coefficients [See Ince, p 534 for references].

In a paper written for the 1748 prize of the Paris Academy, Euler discussed some trigonometry.

> La plûpart du calcul roulera donc sur les angles, que j'introduirai eux-mêmes dans le calcul, en marquaant leurs *sinus, sosinus, tangentes, cotangentes*, par les caracteres *sin, cos, tang, cot* mises devant les lettres qui expriment les angles. Cela abregera très-considérablement le calcul, surtout dans les intégrations et différentiations: or, comme cette maniere d'opérer n'est pas encore reçue généralement, ilsera à propos d'avertir que les différentielles des formules
>
> $$\sin \phi, \quad \cos \phi, \quad \operatorname{tang} \phi, \quad \cot \phi$$
>
> sont
>
> $$d\phi \cos \phi, \quad -d\phi \sin \phi, \quad \frac{d\phi}{\cos \phi^2} \quad \text{et} \quad -\frac{d\phi}{\sin \phi^2} :$$
>
> oú il faut aussi remarquer que $\cos \phi^2$ marque le quarré du cosinus de l'angle ϕ, et $\sin \phi^2$ le quarré du sinus de l'angle ϕ, et not pas le cosinus ou le sinus du quarré de l'angle: ce qui suffira pour l'intelligence des calculs suivans. [3, §14]

> il est à remarquer que $d{\cdot}l\,\mathrm{tang}\,\rho$ signifie la différentielle du logarithme de la tangente de l'angle ρ: ou puisque $d\cdot\mathrm{tang}\rho = \frac{d\rho}{\cos\rho^2}$, on aura, $d\cdot l\,\mathrm{tang}\,\rho = \frac{d\,\mathrm{tang}\rho}{\mathrm{tang}\rho} = \frac{d\rho}{\mathrm{tang}\rho\cos\rho^2} = \frac{d\rho}{\sin\rho\cos\rho}$ 'a cause que ,$\mathrm{tang}\rho = \frac{\sin\rho}{\cos\rho}$. [3, §16]

NB: Put comments about Introductio here.

Chapter 6, "On the differentiation of transcendental functions," of Euler's *Institutiones calculi differentialis* (1755), which is available in an English translation by John Blanton [4] deals with the derivatives — or differentials as he says[7] —of the logarithmic, exponential and trigonometric functions. Euler states that in his *Introductio* (1748) he explained the nature of these functions

> so clearly that they could be used in calculation with almost the same facility as algebraic quantities. In this chapter we will investigate the differential of these quantities in order that their character and properties can be even more clearly understood. [4, p. 99]

The chapter begins with a discussion of the differentials of the logarithm and exponential function. He then cites a formula from from the *Introductio*:

$$\arcsin x = \frac{1}{\sqrt{-1}}\ln(\sqrt{1-x^2} + x\sqrt{-1})$$

whose differential is

$$d(\arcsin x) = \frac{dx}{\sqrt{1-x^2}}.$$

Euler realizes that this seque may be a bit much and so he proceeds to derive the result in another way:

> This differential of a circular arc can also more easily be found without the aid of logarithms. If $y = \arcsin x$, then x is the sine of the arc y, that is, $x = \sin y$. When we substitute $x + dx$ for x, y becomes $y + dy$, so that $x + dx = \sin(y + dy)$. Since
>
> $$\sin(a + b) = \sin a \cdot \cos b + \cos a \cdot \sin b,$$
>
> we have
>
> $$\sin(y + dy) = \sin y \cdot \cos dy + \cos y \cdot \sin dy.$$
>
> As dy vanishes the arc becomes equal to its sine, and the cosine becomes equal to 1. For this reason $\sin(y + dy) = \sin y + dy\,\cos y$, so that $x + dx = \sin y + dy\,\cos y$. Since $\sin y = x$, se have $\cos y = \sqrt{1-x^2}$, from which we obtain
>
> $$dy = \frac{dx}{\sqrt{1-x^2}}.$$

[7] Cite Grabiner's paper. Is this the correct reference: Judith V. Grabiner, The Changing Concept of Change: The Derivative from Fermat to Weierstrass, Math. Mag. 56 (1983), 195-206.

> The arc of a given sine has a differential equal to the differential of the sine divided by the cosine. [4, p. 111]

After deriving the differentials of the other inverse trigonometric functions in similar fashion, Euler proceeds with a series of examples, culminating in

$$\frac{d^{n+1}\arcsin x}{dx^{n+1}} = \frac{1\cdot 2\cdot 3\cdots n}{(1-x^2)^{n+1/2}}\left(x^n + \frac{1}{2}\frac{n(n-1)}{1\cdot 2}x^{n-2} + \frac{1\cdot 3}{2\cdot 4}\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4} + \cdots\right).$$

Euler now proceeds to consider the derivatives of the trigonometric functions:

> Let x be a circular arc and let $\sin x$ denote its sine, whose differential we are to investigate. We let $y = \sin x$ and replace x by $x + dx$ so that y becomes $y + dy$. Then $y + dy = \sin(x + dx)$ and
>
> $$dy = \sin(x + dx) - \sin(x).$$
>
> But
>
> $$\sin(x + dx) = \sin x \cdot \cos dx + \cos x \cdot \sin dx,$$
>
> and since, as we have shown in *Introduction*,
>
> $$\sin z = \frac{z}{1} - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots,$$
>
> $$\cos z = 1 - \frac{z^2}{1\cdot 2} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \cdots,$$
>
> when we exclude the vanishing terms, we have $\cos dx = 1$ and $\sin dx = dx$, so that
>
> $$\sin(x + dx) = \sin(x + dx) + \cos x.$$
>
> Hence, when we let $y = \sin x$, we have
>
> $$dy = dx\cos x.$$
>
> *Therefore, the differential of the sine of any arc is equal to the product of the differential of the arc and the cosine of the arc.*

Euler concludes the chapter with derivations of the differentials of the other trigonometric functions and then a series of examples.

9 Agnesi

In 1749 Jean Jacques d'Ortous de Mairan (1678–1771) and de Montigni reported to the French Academy that Agnesi's second volume, the one dealing with calculus, should be translated into French. They state that the book "contains almost all the discoveries made up to now in differential and integral calculus." They also write that

> It took much skill and sagacity to reduce, as the author has done, to almost uniform methods these discoveries scattered among the works of modern geometers and often presented by methods very different from each other. Order, clarity, and precision reign in all parts of this work. ... We regard it a the most complete and best made treatise ... [**?**, p. 128].

However, a French translation did not appear until 1775 [1]. No translator is named in the volume but Montucla claims that it was Jacques Antoine Joseph Cousin (1739–1800). The 'Advertissement' at the beginning of this edition indicates that only the second volume of Agnesi has been translated. One might surmise that there were ample books in French at this time that dealt with algebra and analytical geometry. The 'Privilége' (p. iv) indicates that the book was examined by d'Alembert, Condorcet and Vandermonde and they gave permission for it to be printed.

At the end of the French edition we find two "Additions de L'Editeur," the first deals with the "Calcul des quantitiés angulaires" (pp. 478–487) and the second is entitled "Remarques sur l'integration des différentielles du premier order." The first of these, on trigonometry, is the more interesting. The author indicates that this was added because Agnesi does not give explicit direct methods of differentiating and integrating the trigonometric functions. I will give one example of each. The author begins by giving ten trigonometric identites which will be used later.

> III. Imaginons que l'angle y augmente de sa différentielle dy; il est clair qu'on aura
>
> $$d(\sin .y) = \sin .(y + dy) - \sin .y,$$
>
> &
>
> $$d(\cos .y) = \cos .(y + dy) - \cos .y$$
>
> ou (Théor. I.),
>
> $$\sin .(y + dy) = \sin .y \cos .dy + \sin .dy \cos .y,$$
>
> & (Théorême III),
>
> $$\cos .(y + dy) = \cos .y \cos .dy - \sin .y \sin .dy.$$
>
> D'un aure côté, l'angle dy étant infiniment petit, on peut supposer dans les deux equations qu'on vient de trouver, $\sin .dy = dy$, $\cos .dy = 1$, comme il est claire pa le série que donne le sinus ou la cosinus par l'angle. Par conséquent, on aura, en subtituant & rédvisant, $d(\sin .y) = dy\ \cos .y$, & $d(\cos .y) = -dy\ \sin .y$.

This derivation follows almost exactly that given by Euler in his *Institutions calcului differentialis* [**?**, §201, p. 116–117], except that Euler explicitly gives the

Taylor series for sin and cos which he truncates to one term. The use of the periods after sin and cos in the French signify that they are abbreviations; this is the same notation that Euler uses in his *Introductio*. There is one significant difference between Euler and Agnesi, however. Agnesi uses the old word ‘angle’, whereas Euler uses the modern, precise term that he himself introduced, ‘arc.’ I suspect that this derivation is taken from Euler, but it is interesting that the more important lesson of using the unit circle has not been leaned.

There are two points dealing with integration in this appendix that are notable. The first involves the integral $\int z\,dz\,\sqrt{(}1+\cos .z)$. We are told to observe that

$$\int z\,dz\,\sqrt{(}1+\cos .z) = x\int dz\,\sqrt{(}1+\cos .z) - \int dz \int dz\,\sqrt{(}1+\cos .z).$$

This formula stumped me at first, for I had never seen a formula when one integrates twice respect to z, but then I realized it is just an integration by parts. Curiously, this integration technique has not been used previously. I know neither the history of this method or where the name comes from.

10 After Euler

Now trigonometry becomes useful. Jean Baptiste Joseph Fourier (1768–1830) completed his important memoir *On the Propagation of Heat in Solid Bodies* in 1807. A committee consisting of Lagrange, Laplace, Monge and Lacroix was set up to report on the work. Lagrange and Laplace objected to Fourier’s expansions of functions as trigonometrical series, what we now call Fourier series. Today this memoir is very highly regarded but at the time it caused controversy, enough so that a revised edition was not published until 1822 as his *Théorie analytique de la chaleur*.

This work of Fourier had an immense impact on both science and mathematics. One of the controversies about Fourier series dealt with their convergence.

Put a bit in about how Cantor’s work on convergence of Fourier series led to set theory.

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