Mystery Circuit in Mathematics Class

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Abstract

We discuss the introduction, student discovery, and teaching of a system of differential equations that model a circuit. In an engineering mathematics course we provide sufficient physics conceptual framework and differential equations background to enable students to model a given two-loop circuit and solve (using Laplace transforms and Mathematica) the system for various values of input voltage. We give each student a unique input voltage frequency and ask for the system response to it. Students pool data, plot the results, and discover that a cascaded high-low filter circuit is the object of their study. We further enhance the modeling perspective by interpreting the system in terms of Laplace transforms and the transfer function for the system. This path permits easy computation of the gain. Students see the meaning of the transfer function and the an example of the power of mathematics in analyzing technology.

Introduction

We present a two-loop circuit to students in an engineering mathematics course. At the time of this activity our students have solved second-order linear constant-coefficient ordinary differential equations (ODEs) by hand, with Mathematica, and by Laplace transforms. Though they have studied fundamental electrical circuit elements in physics classes, we provide a refresher on voltage and current and on the principal devices of capacitor, resistor, and inductor. We direct them to the section of our engineering mathematics text [Kreyszig 1999] for the notions of conservation of charge and Kirchhoff's circuit laws, which help us build differential equations which model our two circuits—one for each loop.

We consider a typical RLC circuit (see **Figure 1**) with resistance R (ohms), inductance L (henrys), and capacitance C (farads or microfarads). An electromotive force $E(t) = E_0 \sin(\omega t)$ (volts) drives the circuit, producing a current, $I(t)$ (amperes).

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Kirchhoff's voltage law says that in the circuit at all times the voltage source $E(t) = E_0 \sin(\omega t)$, equals the sum of the three voltage drops—over R, L, and C, i.e., $E = E_R + E_L + E_C$ (see **Figure 1**). The voltage drop over each of the devices, resistor R, inductor L, and capacitor C, is given by the following formulae, respectively:

(R) $\quad E_R = RI,$

(L) $\quad E_L = LI',$ and

(C) $\quad E_C = \frac{1}{C} \int I(t)\,dt.$

We could review for the students certain physics basics and try to explain the physical devices while deriving these expressions. Usually, though, we take a more accepting view and just use the results to build our differential equations, saving interpretation of the results, (R), (L), and (C), for our analyses of the solutions, as we do below.

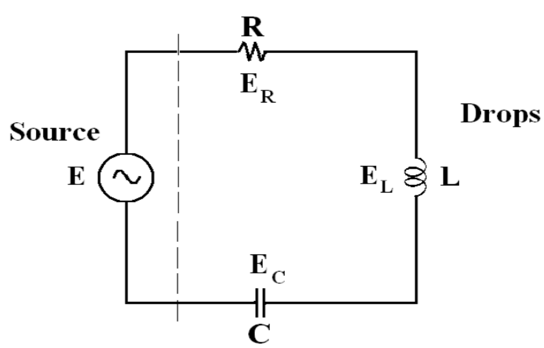

Figure 1. Single loop RLC circuit for student instruction.

Now we are ready to introduce our differential equation in $I(t)$, the current in the one-loop circuit of **Figure 1**, using Kirchhoff's voltage law. We sum the voltages across the resistor, the inductor, and the capacitor and set this sum equal to the voltage $E(t)$ from the driver:

$$LI'(t) + RI(t) + \frac{1}{C} \int I(t)\,dt = E_0 \sin(\omega t), \tag{1}$$

We convert this equation into a second-order differential equation in $I(t)$ by differentiating both sides:

$$LI''(t) + RI'(t) + \frac{1}{C} I(t) = E_0 \omega \cos(\omega t). \tag{2}$$

Prior to the voltage source being applied, there is nothing happening in the circuit, that is, there is no voltage or force to move the electrons in the circuit; hence we have the reasonable initial conditions $I(0) = 0$ and $I'(0) = 0$.

At this point in the course, we often move out into various areas of interest with respect to a single-loop circuit. We introduce terms such as *reactance* and *impedance* and see the roles that these concepts play in the solution. We discuss the system's response to the driver voltage, that is, what the current out (and hence the voltage over a resistor) is for a voltage in; and we consider the notion of *gain* for a circuit. We ask the question: For a given single-loop RLC circuit, what input voltage frequency will give rise to the greatest "output" voltage, i.e., current $I(t)$ and hence voltage over a specific resistor, $RI(t)$? We discuss how a radio "picks" signals out of the ether by varying the capacitance and thereby changing the peak response frequency, which is then amplified for us to hear. Of course, we are attentive to the mathematics; but we find that students are motivated to study the mathematics when they have a setting of interest to them, and circuits provide that.

Course Activity

We describe a course activity that can take place in one to two class periods. We give the class an electrical circuit and each student a unique value of a parameter—the frequency of the driver voltage. We ask each student to model the circuit, solve for the gain associated with the student's assigned parameter value, and report back this gain value. We collect the data, plot it to see what we can learn, and then ask the students to describe what they think this circuit really does. The circuit that we offer our students is shown in **Figure 2**.

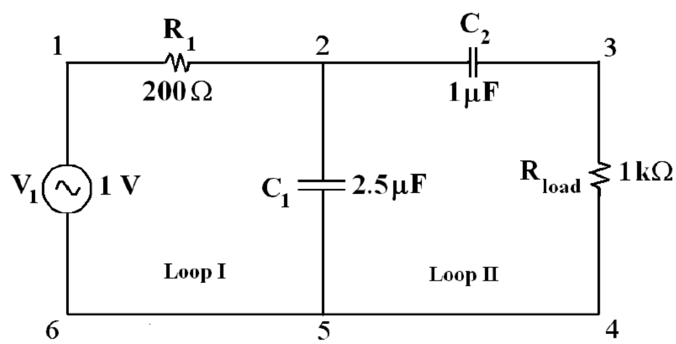

Figure 2. Two-loop circuit for student analysis.

The values for this circuit are $C_1 = 2.5 \times 10^{-6}$ F (farads), $C_2 = 1.0 \times 10^{-6}$ F, $R_1 = 200\,\Omega$ (ohms), and $R_{\text{load}} = 1000\,\Omega$. We use $E(t) = \sin(\omega t)$ V (volts) with $\omega = 100$ (radians/s) as a trial run.

We build equations to describe this two-loop circuit using Kirchhoff's voltage law as above and introducing *Kirchhoff's current law*, which says that the sum of the currents at any node in the circuit must be 0, that is, what goes into the node has to go out without loss or gain.

We identify variables:

$x(t)$ is the current from node 1 to node 2,

$y(t)$ is the current from node 2 to node 5, and

$z(t)$ is the current from node 2 to node 3 to node 4 to node 5.

One way to measure the behavior of this circuit is to compare the amplitude of the "output" voltage $z(t)R_{\text{load}}$ (recall that voltage is current times resistance) with that of the input voltage $E(t)$.

Students can apply Kirchhoff's two laws to find the relationship between the source voltage $E(t)$ in Loop I and the sum of the voltages across the devices in Loop I,

$$\sin(100t) = x(t)R_1 + \frac{1}{C_1}\int y(t)\,dt; \tag{3}$$

and in Loop II the students use the resultant voltage across C_1 as a source voltage and add the voltages across C_2 and R_{load},

$$\frac{1}{C_1}\int y(t)\,dt = \frac{1}{C_2}\int z(t)\,dt + z(t)R_{\text{load}}. \tag{4}$$

Finally, we use Kirchhoff's Current Law at node 2:

$$x(t) = y(t) + z(t). \tag{5}$$

Imposing initial conditions that all currents and all change (derivative) in currents are 0, we have in **(1)–(3)** what we need in order to solve for each of the three currents, $x(t)$, $y(t)$, and $z(t)$. In particular, we can solve for $z(t)$ and determine the amplitude of $z(t)R_{\text{load}}$, the output voltage, to compare to the amplitude of $E(t) = \sin(100t)$, the input voltage. This ratio of output voltage to input or source voltage is called *gain*. We seek the gain for various input voltage frequencies ω with $\omega = 100$ Hz (radians/s) being the trial-run frequency. We recall that the input or source voltage frequency is exactly the same as the output voltage, so in studying gain, we are merely comparing amplitudes of the same frequency voltages.

We then assign each student a unique frequency ω for an input voltage $E(t) = \sin(\omega t)$ and ask for the gain for that frequency. Finally, we plot gain vs. input frequency and see what the students think of such a plot (see **Figure 3**). What does the plot tell us? For this circuit, we assign input voltage frequencies ω in the range $(0, 1000]$.

Laplace Transform Solution Strategy

We demonstrate here the analysis through which we guide our students for $\omega = 100$ to determine the gain of the circuit. We then leave them to analyze the gain from their own input voltage frequency for homework.

We use Laplace transforms in Mathematica to solve this system of differential equations: `LaplaceTransform[f[t],t,s]` transforms $f(t)$ from the time domain t to the frequency domain s.

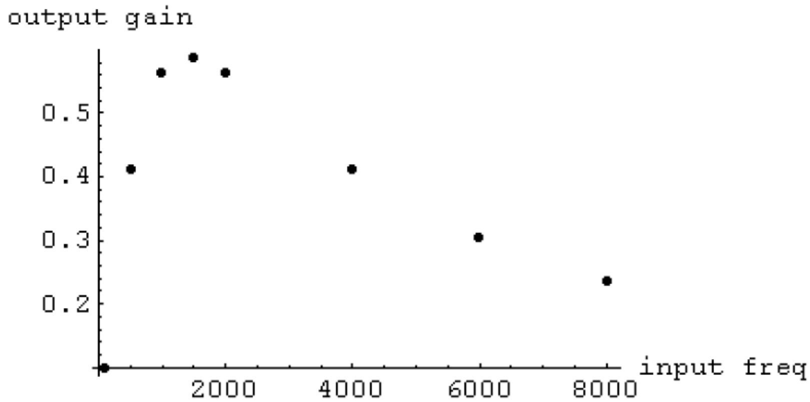

Figure 3. Gain data collected from the student efforts with their assigned input voltage frequencies ω.

There are other approaches that we could use. For example, after differentiating **(3)–(4)**, we can obtain a system of two first-order differential equations, say in $x(t)$ and $z(t)$, by using **(5)** to eliminate $y(t)$. We can then apply Mathematica's `DSolve`:

```
C1 = 2.5/10^6; C2 = 1/10^6; R1 = 200; Rload = 1000;}
DSolve[{100*Cos[100*t] == R1*x'[t] + x[t]/C1 - z[t],}
    (x[t] - z[t])/C1 == z[t]/C2 + Rload*z'[t]}, {x[t], z[t]}, t]}
```

However, as we will see, the Laplace transform approach offers conceptual "gains" to be made.

The Laplace transform $F(s)$ of a function $f(t)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Here are the Laplace transforms of **(3)–(5)**:

$$\frac{100}{10000 + s^2} = 200X(s) + \frac{400000Y(s)}{s}, \tag{3$'$}$$

$$\frac{400000Y(s)}{s} = \frac{1000000Z(s)}{s} + 1000Z(s), \tag{4$'$}$$

$$X(s) = Y(s) + Z(s). \tag{5$'$}$$

Using Mathematica's `Solve` command, we obtain:

$$Z(s) = \frac{200s}{(s + 756.6)(s + 2643.4)(s^2 + 10000)}.$$

The Apart command does partial fraction decompositions and hence offers students convenient and routine access to the transforms of the corresponding transient and steady-state portions of the solutions:

$$Z(s) = \underbrace{-\frac{0.00013769}{s+756.6} + \frac{0.000040046}{s+2643.4}}_{\text{transient solution}} + \underbrace{\frac{0.0000976519s + 0.0016684}{s^2+10000}}_{\text{steady state solution}},$$

From their familiarity with inverting the Laplace transforms of

$$\frac{1}{s+a}, \qquad \frac{1}{s^2+a^2}, \qquad \text{and} \qquad \frac{s}{s^2+a^2},$$

students identify the transient and steady-state Laplace transform pieces (or else they perform the Inverse Laplace transform calculation and make the transient and steady-state identifications in hindsight).

Applying the InverseLaplaceTransform command, we obtain $z(t)$, the current over the resistor R_{load}:

$$\begin{aligned} z(t) \quad &= \quad \underbrace{0.000040046e^{-2643.4t} - 0.00013769e^{-756.602t}}_{\text{transient solution}} \\ &\quad + \underbrace{0.000097652\cos(100t) + 0.000016684\sin(100t)}_{\text{steady state solution}}. \end{aligned}$$

Thus, to obtain the amplitude of the steady state, since the contribution of the transient disappears very rapidly, we use trigonometric identities to combine the sine and cosine terms into one phase-shifted sine term with the amplitude of the current $z(t)$ passing through resister R_{load}:

$$\text{amplitude}_{\text{current}} = \sqrt{0.000097652^2 + 0.000016684^2} = 0.0000990669.$$

This means that the amplitude of the voltage across R_{load} is

$$\text{amplitude}_{\text{voltage}} = \text{amplitude}_{\text{current}} \cdot R_{\text{load}} = 0.0000990669 \cdot 1000 = 0.0990669\ .$$

Now, the gain for the chosen frequency $\omega = 100$ is the amplitude of the voltage across the resistor R_{load}, which is 0.0990669, divided by the amplitude of the input voltage $E(t)$, which is 1, so

$$\text{gain} \quad = \frac{0.0990669}{1} = 0.0990669.$$

Student Activity

At the next class, we ask for a report from each student: "What was your assigned frequency and what gain did you obtain?" We plot these roughly on the board before inputting them as data pairs into Mathematica and using `ListPlot` (see **Figure 3**). Incidentally, errors in student work are glaring when each individual student's data point is plotted in this manner!

Once we all see this plot, we know something about what this circuit does. It appears to have low gain for both very low- and very high-frequency input voltages. Almost always, there is a student who remarks that this is the characteristic of a *filter*, that is, the circuit takes all the middle-frequency input voltages and keeps their gains high while dropping the gains for very low and high input voltage frequencies. Indeed, we ask the students to change their ω to either a very high or a very low frequency and compute the new gain. These results add points at the left and right sides of the plot in **Figure 3** and confirm their interpretation of the plot.

A further activity is to ask the students to consider several variations on the source voltage, perhaps a sum of different frequencies as a single source voltage, e.g.,

$$E(t) = \sin(200t) + \sin(2000t) + \sin(20000t),$$

where the respective gains of the components are 1.92×10^{-4} for $\omega = 200$, 5.64×10^{-4} for $\omega = 2000$, and 0.99×10^{-4} for $\omega = 20000$. Clearly, the low- and high-frequency components have lower gain than the mid-frequency one!

Transfer Functions in Laplace Transforms

One of the objectives in our engineering mathematics course is when possible to use the vocabulary that our students will encounter in their prospective engineering disciplines. The notion of the transfer function, indeed, of living in the frequency domain of the Laplace transform, is something that we discuss when we do second-order linear constant-coefficient ODE work earlier in the course. Now that concept pays off as we try to analyze what is going on in this circuit.

For a differential equation in $y(t)$ with driver $f(t)$, and zero for the initial conditions,

$$a\,y''(t) + b\,y'(t) + c\,y(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{6}$$

the *transfer function* $T(s)$ is the ratio of the Laplace transform $Y(s)$ of the solution to the Laplace transform $F(s)$ of the driver, or

$$T(s) = \frac{Y(s)}{F(s)}.$$

This is almost like our notion of gain, ratio of "out" to "in," but not quite.

We show how this all fits together in the situation of **(6)**. Taking the Laplace transform of both sides and incorporating the initial conditions gives

$$(as^2 + bs + c)Y(s) = F(s),$$

so

$$Y(s) = \frac{1}{as^2 + bs + c} F(s). \tag{7}$$

and

$$T(s) = \frac{1}{as^2 + bs + c}.$$

We return to our two-loop circuit. We let $U(s)$ denote the Laplace transform of $E(t)$, the source voltage, and $Z(s)$ denote the Laplace transform of the output voltage, so that $T(s) = Z(s)/U(s)$.

If we multiply the output transform by a resistance, R_{Load}, we can get the Laplace transform of the output voltage using Ohm's Law $E = IR$.

If we consider the amplitude of each of the transforms $U(s)$, $T(s)$, and $Z(s)$, we see that the amplitude of $T(s)$ is

$$\frac{\text{amplitude of } Z(s)R_{\text{load}}}{\text{amplitude of } U(s)}.$$

Using $i\omega$ for s will permit us to understand the amplitude (or gain) ratio at a particular frequency ω, while plotting $|T(i\omega)|$ permits us to see the gains for all values of ω.

We explain the previous paragraph in a more general setting—and thereby show the power of the transfer function—by examining a general second-order linear constant-coefficient ODE,

$$ay''(t) + by'(t) + cy(t) = f(t),$$

with driver $f(t) = C\sin(\omega t)$ and initial conditions $y(0) = 0$ and $y'(0) = 0$. The solution has a transient portion, which in our case will go to 0 quickly, leaving only the steady-state portion of the solution, namely $A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)$, where the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ ($A(\omega)$ and $B(\omega)$, respectively) are real functions of ω. Now $y(t)$ and $A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)$ differ, after a very short time, by a negligible transient solution, thus we write

$$y(t) \sim A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t).$$

So let us compute the Laplace transform of all the terms in the steady-state solution only and examine the transfer function as it applies to the input voltage only:

$$F(s) = \frac{C\omega}{s^2 + \omega^2} \quad \text{and} \quad Y(s) = \frac{A(\omega)s}{s^2 + \omega^2} + \frac{B(\omega)\omega}{s^2 + \omega^2}.$$

The transfer function is then

$$T(s) = \frac{Y(s)}{F(s)} = \frac{\frac{A(\omega)s}{s^2+\omega^2} + \frac{B(\omega)\omega}{s^2+\omega^2}}{\frac{C\omega}{s^2+\omega^2}} = \frac{A(\omega)s + B(\omega)\omega}{C\omega}.$$

We substitute $s = i\omega$ into $T(s)$ and see what happens. We have

$$T(i\omega) = \frac{A(\omega)i\omega + B(\omega)\omega}{C\omega} = \frac{A(\omega)i + B(\omega)}{C}$$

and we obtain

$$|T(i\omega)| = \frac{\sqrt{A(\omega)^2 + B(\omega)^2}}{C}, \tag{8}$$

since the terms $A(\omega)$, $B(\omega)$, and C are real. We see that $|T(i\omega)|$ gives the amplitude of the steady state output function $y(t) = A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)$ divided by the amplitude of the input function $f(t) = C\sin(\omega t)$. This ratio is simply the gain. This is an "Aha!" moment, for we see that by taking the absolute value (magnitude) of the transfer function, $T(s)$, with $i\omega$ substituted for s we easily produce the gain *for any* ω.

Returning to the case of the circuit, Mathematica can show that the transfer function $T(s)$, which is the Laplace transform of the output voltage across R_{load}, i.e., $Z(s) \cdot R_{\text{load}}$, divided by the Laplace transform of the input function $u(t) = \sin(\omega t)$, i.e., $U(s) = \omega/(s^2 + \omega^2)$, is simply

$$T(s) = \frac{2000s}{2 \times 10^6 + 3400s + s^2}.$$

When we substitute $s = i\omega$ and take the absolute value of $T(i\omega)$, we obtain

$$\text{gain}(\omega) = \left| \frac{2000\omega}{2 \times 10^6 + 3400i\omega - \omega^2} \right|. \tag{9}$$

We offer a plot of $\text{gain}(\omega)$ vs. ω in **Figure 4** and then a plot of our student-computed data over the theoretical gain (see **Figure 5**).

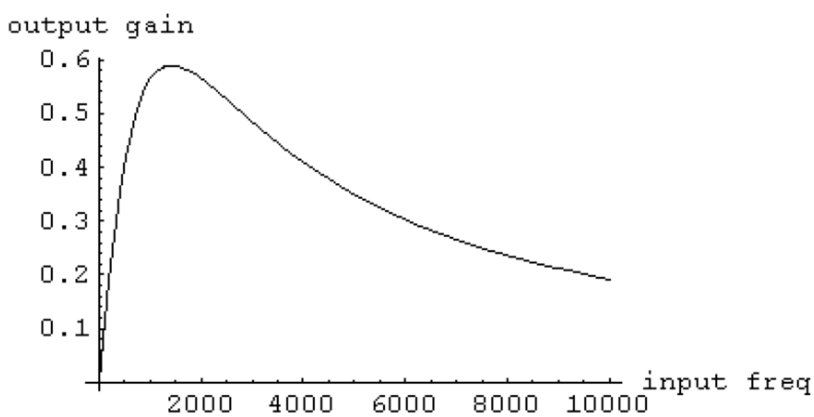

Figure 4. Plot of gain(ω) for our two-loop circuit.

A picture is worth a thousand words, and so the students see the value of the transfer function in easily computing the gain for the circuit. They can also see qualitatively that $\text{gain}(\omega)$ gets small as ω increases, since the denominator

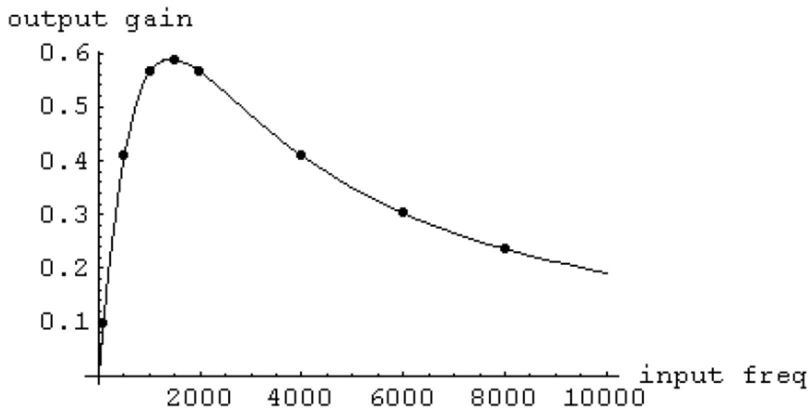

Figure 5. Plot of gain(ω) and student data for our two-loop circuit.

of **(9)** has a term containing the square of ω while the numerator contains ω only linearly. As ω gets small, they can see that the numerator goes to 0 while the denominator goes to 2×10^6, hence $\text{gain}(\omega)$ goes to 0. Thus, the gain on either end, ω small or big, is small; so this circuit keeps the amplitude of midrange frequencies high while lowering—considerably—the amplitude of the lower- and higher-range frequencies: It is indeed a filter!

Conclusion

We have described how we use simple laws of circuits (Kirchhoff's laws) to build differential-equation models, with an example of a two-loop circuit for which students plot pooled data to discern the nature of the circuit.

Further, we have shown how we use the concept of transfer function in the Laplace transform solution strategy to compute the gain of a circuit as a function of the input voltage frequency.

All of this is in keeping with our attempt to relate the mathematics in our engineering mathematics course to the engineering that our students study and to show them the value of sophisticated tools—in mathematics and in Mathematica—for their future.

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