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Seeing around a ball: complex, technology-based problems in calculus with applications in science and engineering-redux

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year in which it was drawn? (The actual year of publication is the same as that of the *King James* or *Authorised Version* of the bible.)

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Seeing around a ball: complex, technology-based problems in calculus with applications in science and engineering – redux

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A complex technology-based problem in visualization and computation for students in calculus is presented. Strategies are shown for its solution and the opportunities for students to put together sequences of concepts and skills to build for success are highlighted. The problem itself involves placing an object under water in order to actually see more of the object due to the refraction of light.

1. Introduction

A number of years ago we developed a set of complex, technology-based problems in calculus with applications in science and engineering. A number of our efforts can be found in the form of completely posed and worked out problems at http://www.rose–hulman.edu/Class/CalculusProbs/[1].† We have shared our enthusiasm for these problems [2] in which there are a variety of approaches to the problem, but they usually demand bringing together a number of concepts and skills, hence complex; moreover, technology is a very helpful, often necessary tool.

We are always on the lookout for such problems, for scenarios in which to develop them, and we continue to use them in our teaching [4]. In [3] we offer a problem in visualization which uses multivariable calculus concepts. The problem is essentially to describe (mathematically) what we can see on one mountain while sitting on an adjacent mountain. See Figure 1. This leads to a complex problem in which technology is necessary to offer a full analysis.

In this note we consider an illustration of this kind of problem in optics. We take the reader through the presentation of the problem, strategies for doing the problem, and discussions of pedagogical and technological issues.

2. Seeing around a circular disc under water

We all have some experience with seeing objects which are under water from above the surface of the water. They are not where they appear to be and we find this out by reaching to where we see them, only to find them not there! We consider the situation of looking directly down at a sphere while our eye is above the surface of the water and the sphere is under the water. Our eye is directly above the North Pole of the sphere and we are interested in seeing exactly how much of the sphere we can see when it is under water and comparing that to how much of the sphere we can see when there is no water. Perhaps a better way to introduce students to this problem is to ask them if they can see more, less, or the same amount of the sphere's surface in the two situations. We can develop some conjectures in class and small group discussions and then proceed to analysis to determine just what portion of the surface of the sphere we can see in the two situations.

We first consider this situation in one less dimension, i.e. we consider the setup (see Figure 2) in which our eye is directly above the centre of a circular disc (referred to as circle) and in the plane of the circle which sits perpendicular to and on the bottom of a tank of water. We seek to compare the two situations – with and without

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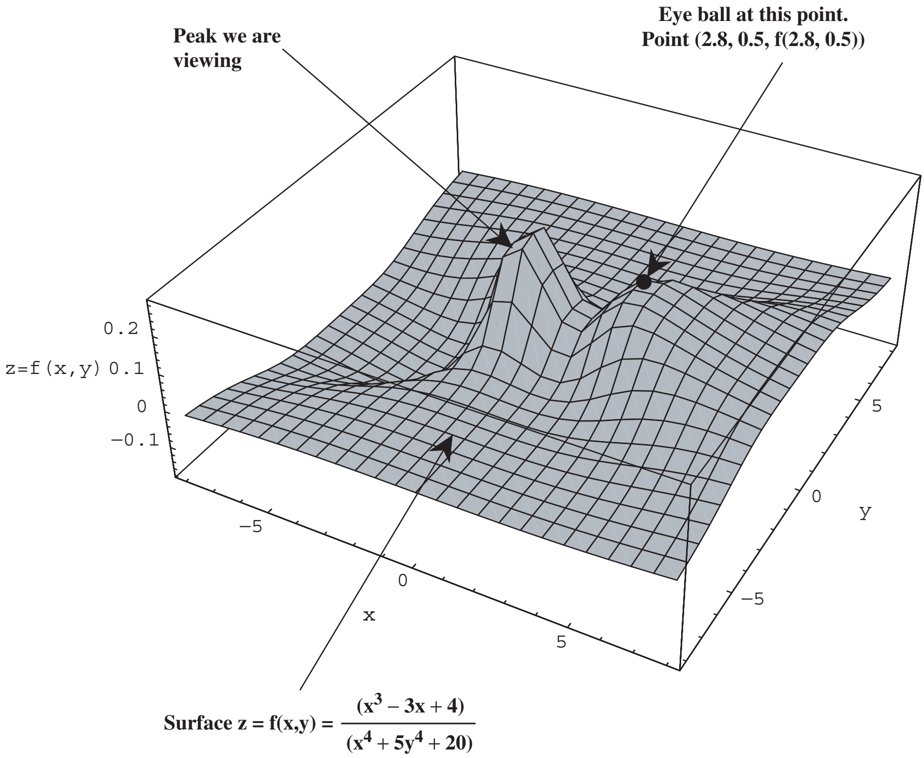

Figure 1. Surface $z = f(x, y) = (x^3 - 3x + 4)/(x^4 + 5y^4 + 20)$.

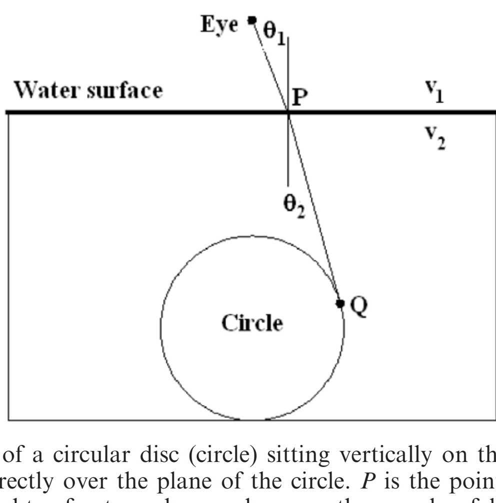

Figure 2. Depiction of a circular disc (circle) sitting vertically on the bottom of a pool of water with our eye directly over the plane of the circle. P is the point on the surface of the water at which the light refracts and v_1 and v_2 are the speeds of light in air and water, respectively.

water – and determine how much of the circumference of the circle we can see in each situation. To move to the spherical situation we rotate our circumference elements about the vertical diameter of the circle to compute the visible surface areas and compare. Thus we settle in, for now, to the problem of the vertical circle.

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Light travels from Q, the point we "see" on the circumference of the circle, to our eye, Eye (see Figure 2). However, the light bends in such a way that the path obeys Snell's Law for refraction. This law says that

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}$$

or

$$v_2 \sin(\theta_1) = v_1 \sin(\theta_2)$$

where θ_1 is the angle between the line of vision from Eye to P and the vertical line to the flat surface of the water at the point, P, where the refraction or bending of the light occurs, and θ_2 is the angle between the line of light from P to the point Q on the disk we "see" and the vertical line to the flat surface of the water at the point, P.

We can obtain Snell's Law, a relationship between the speed of light in one medium and an adjacent medium and the angle of incidence of light and the angle of refraction of light at the interface of the two media, through an optimization process by using Fermat's Principle which says that light travels from point Q to the Eye along the path of least time. This can be a separate activity or a preliminary part of this larger problem. The derivation of Snell's Law from Fermat's Principle is a standard calculus text exercise. We work from Snell's Law in this article.

Now if we emptied all the water out of the tank and the circular disc were in the same position, then our line of vision from Eye to a point of tangency, call it Q', on the circle, would be just a small distance up the circle from Q. Incidentally, in Figure 2, the line from P to Q has to be tangent to the circle at Q as well. Figure 3 shows the refracted light path in the case where there is water present and the straight line path of light (to the left of the bent path through P) if there is only air present.

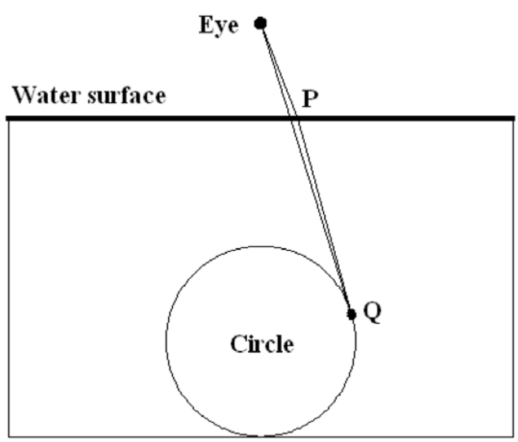

Figure 3. Depiction of a circular disc (circle) sitting vertically on the bottom of a pool of water with our eye directly over the plane of the circle. Two paths of light from Q to the Eye are shown (1) the refracted light through the point P on a bent path from Q to the Eye and slightly to the left of this bent path we see (2) the straight path of light if there was no water, only air. This latter light emits from a point Q' slightly up the circle from point Q. See Figure 4 for close up.

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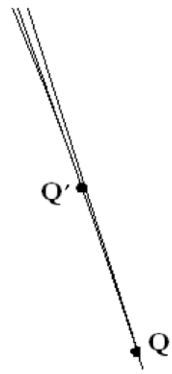

Figure 4. Close-up of the circle. Point Q' is the point we can see from Eye if there were no water while point Q, due to the refraction of the light in water and hence the bending "out" of the light away from the circle, is the point we could see if water were present. Notice the two tangent lines to the circle at Q' directly from Eye and at Q from point P, where the light is refracted at the water's surface.

If we were to zoom in around the point Q' we would see that because of the refraction of light due to the water and hence the bending "out" of the light away from the circle we can actually see farther along the circumference of the circle to point Q, than to point Q'. See Figure 4.

Now with this introduction for the reader, here is the question we ask our students after this initial discussion and conjecturing.

> "How much more of the circumference of the circle can we see when we submerge it into water and the geometry remains the same?"

This is the challenge. This is the complex problem and technology that will help us solve it.

2.1. *Straight light path – no water, circle in air*

We first consider what has to be done, i.e. what mathematical formulations can be rendered, to determine just how much of the circumference of the circle we can see if there is no water. We do this in a classroom setting in which students work in small groups and have access to blackboards at which they sketch out their ideas and computers where they can implement ideas and get quick feedback. We seek to determine the extreme point(s) that can be seen on the circumference of the circle from Eye. From these we can compute the portion of the circumference of the circle we can see. By symmetry we focus our analysis on one side of our circle and double our results as appropriate. See Figure 5.

We give coordinates to our situation. Suppose the origin $(0, 0)$ is at the bottom of the circle – the base of the tank. The horizontal line along the base of the tank will be the x-axis while the vertical line from the point $(0, 0)$ through the center of the circle (a, b) to our Eye will be the y-axis. The equation of the circle with center $O = (a, b)$ and radius r is $(x-a)^2 + (y-b)^2 = r^2$. We seek to find the coordinates, (u, v) of the point Q' on the circumference of the circle that is the extreme (farthest right) point we can see from Eye at $(0, s+h)$ where s is the depth of the water when we do fill the

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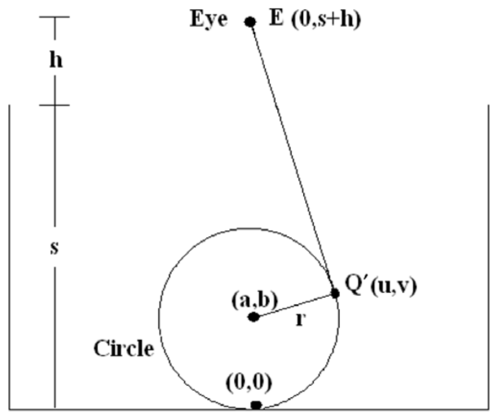

Figure 5. View of Eye–Circle configuration showing our labels and the extreme (right) point Q' we can see without the presence of water.

tank and h is the height of the eye above the surface line of the water. Obviously, the line from Eye at $(0, s+h)$ to Q' is a tangent line, and as such, this line, call it $\overline{EQ'}$, is perpendicular to the radius of the circle. In this case we have a vector argument that the tangential vector $\overline{EQ'} = \langle u-0, v-(s+h)\rangle$ is perpendicular to the radial vector, $\overline{OQ'} = \langle u-a, v-b\rangle$. Using the fact that perpendicular vectors have their dot product equal to 0 we have our first equation or constraint on the point $Q' = (u, v)$:

$$\big\langle u-0, v-(s+h)\big\rangle \cdot \langle u-a, v-b\rangle = 0. \tag{1}$$

The second constraint is the fact that the point $Q' = (u, v)$ lies on the circle:

$$(u-a)^2 + (v-b)^2 = r^2. \tag{2}$$

Now, as a teacher, one is always concerned with how we are to proceed and how general the analysis could be. For the purpose of this exploration we use numbers to give students a sense of the situation. We use the following values:

Height of the waterline $s = 10\,\text{m}$,
Distance to the Eye above the waterline $h = 2\,\text{m}$,
Radius of circle $r = 3\,\text{m}$, and
Centre of circle $(a, b) = (0, 3)\,\text{m}$.

In Mathematica we offer the following command to solve the two equations for the two unknowns u and v:

$$\text{Solve}[\{\{u-0,\ v-(s+h)\}\cdot\{u-a,\ v-b\} == 0,\ (x-a)^2+(y-b)^2 == r^2\}, \{u,\ v\}]]$$

We obtain two solutions. From the form of the solution students can tell that one solution corresponds to the x-coordinate of the extreme point Q' we can see on the right side of the circle (Figure 5) while the other solution corresponds to the x-coordinate of the extreme point we can see on the left side of the circle. There is

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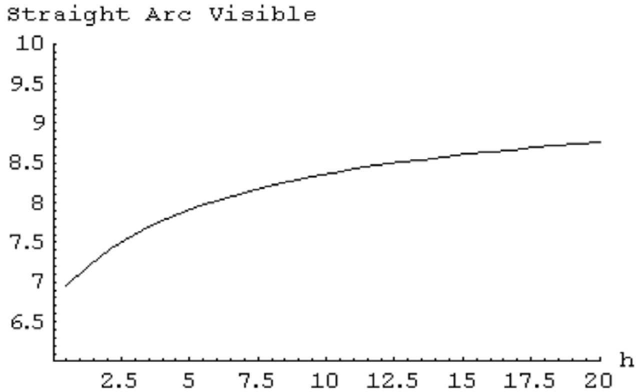

Figure 6. For the case of no water this is a plot of the amount of the circumference (radians) that can be seen with the eye at height h above the surface of the water, directly above the centre of the circle. Notice how this approaches 3π (half the total circumference of the circle) as h increases.

symmetry about the vertical line from our Eye to the centre of the circle and we can confine ourselves to the computations with Q' and take advantage of the symmetry when appropriate. Thus, our point $Q' = (u, v)$ is given in terms of h by:

$$u = u(h) = \frac{3\sqrt{(h+4)/(h+7)}}{\sqrt{(h+7)/(h+10)}} \quad \text{and} \quad v = v(h) = \frac{3(h+10)}{h+7}.$$

We do a reality check at this point. As $h \to \infty$, i.e. we are moving farther away from the circle and we can see more and more of the circle's circumference, indeed, "at" $h = \infty$ we can see half of the circle for

$$\lim_{h\to\infty} (u(h), v(h)) = \lim_{h\to\infty} \left(\frac{3\sqrt{(h+4)/(h+7)}}{\sqrt{(h+7)/(h+10)}}, \frac{3(h+10)}{h+7} \right) = (3, 3)$$

and $(3, 3)$ is the extreme point on the horizontal diameter through the point $(a, b) = (0, 3)$.

With a bit of geometry and trigonometry we can compute, as a function of h, the amount of the circumference in metres we can see when our Eye is placed at height h m above the surface of the water, directly above the centre of the circle. We actually double the portion of the circle we can see on the right side. We call this function see(h).

$$\text{see}(h) = 2\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{v(h) - 3}{u(h)} \right) \right).$$

We offer a plot of see(h) versus h in Figure 6 and note the asymptotic behavior of the amount of the circle we can see getting closer to 3π as h increases.

2.2. *Bent light path – circle in water*

Now let us turn to the situation in which we have water in the tank to a depth of $s = 10$ m, i.e. covering the disc. This is the situation in Figure 2. We are

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given the location of the our Eye at $(0, s+h)$, h m above the height of the water, which is s m deep.

Using the same coordinate system as in the situation in which there is no water we seek to find the path of light from our Eye $(0, s+h)$ to the new point of tangency, $Q=(u, v)$, as the ray of light crosses the interface between air and water at some point, we called it $P=(x, s)$. Here, x m is the distance along the water's surface from the vertical line from our Eye through the centre of the circle to the point P at which the light path cuts the air–water interface in Figure 2. We now have three constraints on our variables

$Q=(u, v)$ must be on the circle. This yields equation (3):

$$(u-a)^2+(v-b)^2=r^2. \tag{3}$$

The line $\overline{PQ}$ must be tangent to the circle, i.e. $\overline{PQ}$ must be perpendicular to the radial vector connecting (u, v) and the centre of the circle (a, b). This means the dot product of $\langle u-x, v-s\rangle$ and $\langle u-a, v-b\rangle$ must be 0. Hence equation (4):

$$\langle u-x, v-s\rangle \cdot \langle u-a, v-b\rangle = 0. \tag{4}$$

Finally, we use Snell's Law, $v_2 \sin(\theta_1)=v_1 \sin(\theta_2)$, where θ_1 is the angle between the vertical line through our Eye and the line from our Eye to the point, $P=(x, s)$, and θ_2 is the angle between the vertical line through our Eye and the line of light from P to the point $Q=(u, v)$, to give a third constraint in equation (5).

$$v_1 \frac{u-x}{\sqrt{(u-x)^2+(s-v)^2}} = v_2 \frac{x}{\sqrt{h^2+x^2}}. \tag{5}$$

In the case of no water we were able to determine the coordinates of the extreme point on the circle we could see as an explicit function of h and from this compute the amount of the circumference of the circle we could see as a function of h, see(h). However, in this case equations (3), (4), and (5) yield no analytic solution for the coordinates u and v as a closed-form function of h. Thus, we need to determine the point Q and from this the amount of the circumference of the circle we can see for specific numerical values of h. Mathematica permits us to build a list, CrookedData, of the data $(h, u(h), v(h))$ by marching through values of h from, say, $h=0.5$ to $h=20$ in steps of 0.5 while Appending the data triples as we go. The absolute values (Abs) are to ensure that we get the data points on the right side of the circle. The [[2]] forces Mathematics to pick the second of the two solutions in sol./. is Mathematica's way of saying, "according to" the second solution (sol [2]).

```
CrookedData={};
Do[sol=NSolve [{eq1, eq2, eq3}, {x, u, v}];
  AppendTo [CrookedData, {h, Abs [u/. sol [[2]]], Abs [v/.sol
[[2]]]}], {h,.5, 20,.5}]
```

Now we fit an interpolating function through this data and plot this new function of h, which computes the amount of the circumference of the circle we can see with water present. We use Mathematica's InterpolationFunction command. In Figure 7 we plot this numerical function (thick) with the analytic function obtained in the case when no water was present (thin).

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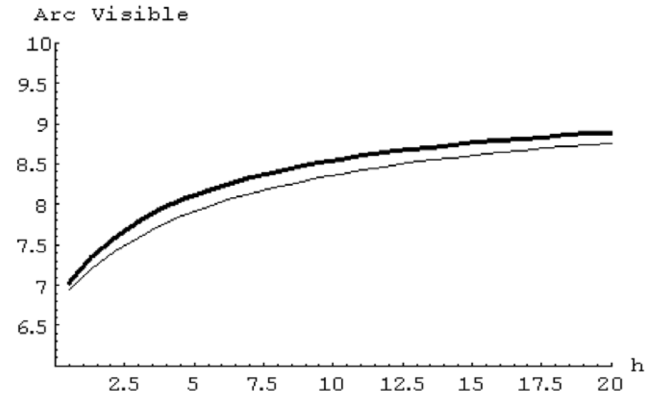

Figure 7. Plots of amount of the circumference of the circle we can see with water (thick) and no water (thin).

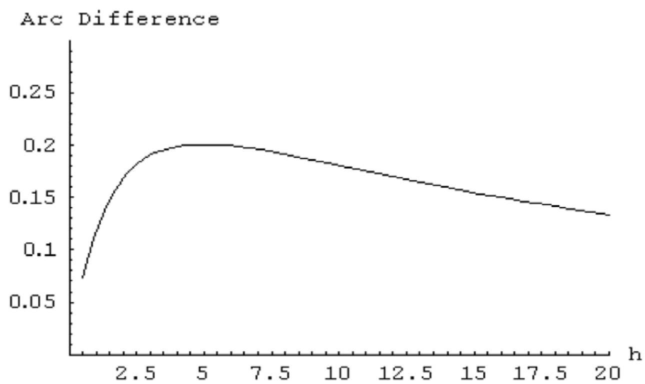

Figure 8. Plot of the difference in the amount of the circumference of the circle we can see with water and without water.

Figure 7 shows that we can see more of the circumference with water present (thick line). This is not surprising. How much more arc of the circumference of the circle can we see at each new height h? In Figure 8 we plot the difference between the two functions.

With what we have done so far here is another question we ask:

> "At what height, h, for our eye above the surface of the water, is the increase in the circumference of the circle we can see between the two cases – without and with water – greatest?"

From the two functions we have for the amount of circumference of the circle (radius $r = 3\,\mathrm{m}$), one with water and one without water, we can compute the maximum difference and we find it to be $0.200659\,\mathrm{m}$ at a height of $h = 4.9745\,\mathrm{m}$. This is an increase of 1.0645% in the amount of the circle visible due to the presence of water. This tells us that not only can we see more when water is present, but there is a height at which our difference is greatest.

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3. Seeing around a ball under water

We now return to our original problem of looking directly down at a sphere while our eye is above the surface of the water and the sphere is under water. Again, our eye is directly above the North Pole of the sphere. We are interested in seeing exactly how much of the sphere we can see when it is under water and comparing this with the amount we can see if there is no water.

We represent the circle of Figure 5 parametrically by $x(t) = a + r\sin(t)$ and $y(t) = b + r\cos(t)$, $0 \le t \le 2\pi$. We use this to compute the surface area swept out by a piece of this circular arc from an angle 0 to an angle β (both, measured off the vertical line from our Eye to the centre of the circle (a, b)) on the right side of the circle as seen in Figure 5 as this piece of arc rotates about this vertical line:

$$\int_0^{\beta} 2\pi x(t)\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t.$$

We use this to do our computations of how much of the surface of the sphere we can see both with and without water present. We build two functions for the amount of the surface of the sphere we can see in these two instances, both as a function of h.

From the two functions we have for the amount of surface area of the sphere of radius $r = 3\,\mathrm{m}$ (with and without water) we can see in terms of h we plot the difference in the amount of the surface of the sphere we can see with water and without water as a function of h see Figure 9. From this we compute the maximum difference in the surface area of the sphere we can see and we find it to be $1.83986\,\mathrm{m}^2$ when our eye is at a height of $h = 5.27441\,\mathrm{m}$. This is an increase of 1.6268% in the amount of the sphere visible due to the presence of water. Again, this tells us that not only can we see more of the sphere when water is present, but there is a height at which our difference is greatest. Additionally, the maximum percent difference in the surface area we can see is about a half a percent more than in the increase we obtained for the maximum percentage difference in the amount of circumference we can see.

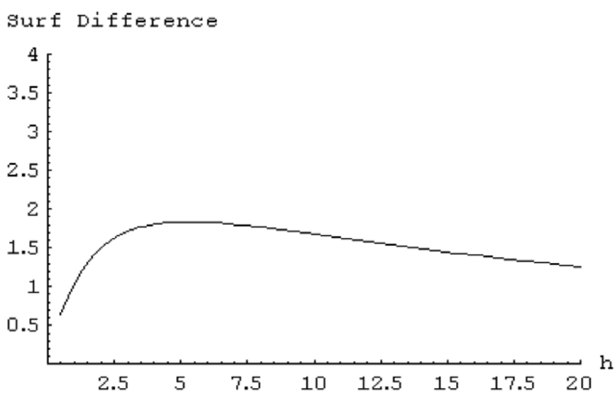

Figure 9. Plot of the difference in the amount of the surface of the sphere we can see with water and without water.

4. Conclusion

We have presented a calculus problem with some nice geometry, trigonometry, and optics, which is complex, benefits from technology, and offers an interesting observation, namely, when we place a sphere in water we can see more of it than when it is in air!

We believe such problems are appropriate to help students put together their studies from different periods in their mathematical education and to make appropriate use of technology. We encourage you to consider this problem and to design others like it for your students so they can experience complex, technology-based problems in calculus with applications in science and engineering.

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We appreciate conversations with our colleague, Frank Wattenberg, concerning the ideas in this note.

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Estimation of half-life for single compartmental elimination models

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A method is presented to calculate accurate approximations to the half-life values of elimination systems modelled by one compartment. The major advantage of this method is that only algebraic mathematical operations are required. The results will be of value not only to students beginning the study of

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