EDITOR'S NOTES

This issue addresses the concept of utilizing projects in the classroom, in particular, projects that are not solely pure mathematical exercises but interdisciplinary projects, which require some research outside mathematics. The question is in what way are the projects a learning tool, and what can be done to improve them? Read these articles to find the answers.

Physics, environmental science, economics, civil and mechanical engineering and ophthalmology are just some of the disciplines that have been used for projects. That our students are seeing mathematics in action is clearly demonstrated in the pages that follow. Not only are the students using the mathematics from the course they are taking but also from previous courses. Great care has been taken by the authors of the projects to link together what was learned before, what is being taught at that time and the subject material of the project.

You will read that not only have our students been enriched by these projects, but our instructors have benefited as well. "There are several mines I have blissfully tripped as I have strolled down the project-writing path… I have assigned one project which achieved a few goals (none of which I had intended)…" (From **"A Project that Worked (?)"**); "This was one project I enjoyed grading." (From **"The Heat Equation without Heat"**); "I am happy to report that he is planning on a double major in mathematics and computer science." (From **"Corneal Topography"**). These are just several notable quotes; there are more inside…read on.

Finally, I would like to thank all of the authors and assistant editors for their help. Thanks to MAJ. Gerald Kobylski for patiently assisting with typesetting.

Best wishes from West Point,
Mary Jane Graham

CONTENTS

SUBSCRIPTIONS TO *MATHEMATICA MILITARIS*:

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The Heat Equation without Heat

MAJ. Kellie A. Simon, USMA

Open a text on Partial Differential Equations (PDE), and more than likely, the first application for the heat equation you encounter will be heat conduction in a long, thin bar. While you probably have fond memories of this classic, I wanted to bring the heat equation to life for students in my introductory PDE course in an application without heat. I developed an out-of-class, semester-long project for my students about underground contaminant flow that worked well, with both intended and unintended positive results.

The students in my course had completed basic studies in ordinary differential equations and multivariable calculus courses, and were just beginning work in their engineering disciplines. Most of the students were environmental engineering majors, so the choice of an environmental application was natural. The application I developed involved a landfill leaking contaminants near a town. Since underground contaminant flow (in the most simplified scenario) obeys the laws of diffusion, I presented the heat equation by its other name, the diffusion equation.

The first part of the project involved background investigations of the situation, in which students developed a 3-D model of an access road to the landfill, and calculated streamlines and flux at monitoring wells near the landfill. The second part of the project involved modeling the contaminant flow out of the landfill. They found the analytic solution of the steady state (2-D) diffusion [Laplacian] equation using separation of variables, and evaluated "safe" areas for future development. In the third part of the project, I changed the physical problem by constructing a wall to stop the contaminants from leaving the landfill and by introducing the flow of groundwater in the region. The steady state conditions from part two became the initial conditions for part three, and the students solved the 1-D advection-diffusion equation (shown below) numerically using finite difference methods.

$$D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} = \frac{\partial C}{\partial t}$$

The project was successful on several counts. First, the students gained a solid grasp on the meaning of boundary conditions. They were able to read a physical description and translate that into the appropriate mathematical conditions. Second, the students understood and used fairly difficult mathematical techniques. They did not merely crunch through an algorithm; they had to determine which mathematical tools were appropriate, and interpret the results within the context of a real-world problem. Third, student feedback on this project was very positive. They were interested in the solutions, and surprised by the relevance and usefulness of the mathematics. They worked hard to grasp even the subtle issues and spent time (especially for the numerical portion) experimenting with the effects of changing boundaries and coefficients. This was one project I enjoyed grading.

I've thought about why this project was successful. Foremost, I believe it was because students understood the problem intuitively. Students understand diffusion: they have seen ink stains spreading on clothing, sugar dissolving into water, and smoke disappearing into thin air. Thus, the application was accessible to all students, and they could "check" their solutions by just thinking about what <u>should</u> happen. For some students, this was a new concept: “Does my answer make sense?” Since students felt comfortable with the problem, they were more confident, and more curious, and more critical of their own solution techniques.

The second reason I believe the project was successful is realism. I know environmental engineers would roll their eyes when I use "real" to describe the sanitized and idealized natural conditions I imposed. In fact, I was very concerned that the problem was too contrived. This, however, was one of the unintended and very positive results of the project. Yes, the students recognized the obvious fallacies: perfectly flat soil layers, homogeneous soil conditions, microorganisms "eating" the contaminants on the edges of a region. Rather than being a distraction, the fallacies spurred student analysis! They identified the assumptions of the model, evaluated the validity of the assumptions, and analyzed the errors caused by those assumptions. The project had realism, but not too much realism.

Studying the heat equation without heat works. My students acquired the concepts and skills, and more importantly, cared about the acquisition. This project provided a good venue for students to understand conditions, model behavior, use mathematics, and analyze results. Next time you reach for that old chestnut, "a long, thin bar", think again.

Making Money with Mathematics

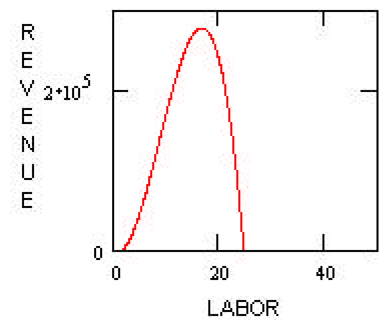
MAJ Gerald C. Kobylski (Department of Mathematics)
MAJ Jim Matheson (Department of Mathematics)
MAJ Jeff Angers (Department of Social Sciences)
United States Military Academy

This past year we had a tremendous experience in developing an Interdisciplinary Lively Applications Project, or ILAP, between the Department of Mathematics and the Department of Social Sciences. In this paper we will discuss the economics application problem, some of the things that went extremely well during the development and implementation of the project, and those things that can be improved by someone implementing a similar project.

The two courses involved in this ILAP were MA205, Multivariable Calculus, and SS201, Economics. MA205 is the third course in a four-course core sequence in the Department of Mathematics. MA205 follows Discrete Dynamical Systems and Calculus and Differential Equations. SS201 is a required course providing an introduction to macroeconomics and microeconomics.

In this project cadets were required to apply concepts from discrete dynamical systems, single variable calculus, and multivariable calculus, in order to model, analyze, and optimize production within a simulated manufacturing company in the short run, the long run, and the very long run. There were four requirements in this project: Requirement I – short run production decisions using single variable calculus, Requirement II- long run production decisions using discrete dynamical systems, Requirements III & IV – long run production decisions and very long run production decisions using multivariable calculus.

In the first requirement, the cadets were given production, revenue, and cost functions. For

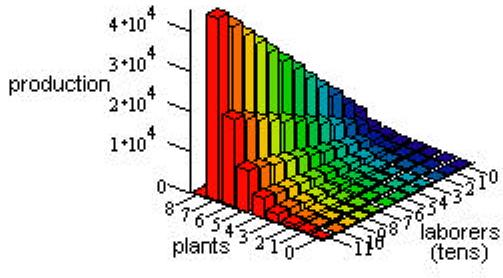


example, here is a typical revenue function graph.

In the short run economic model, labor is the only input for production that varies.

The cadets were required to graph these equations and use single variable optimization techniques to explore the maximum profit and the profitability range of the simulated company. We also required the cadets to use the cost and revenue equations along with their knowledge of single variable calculus optimization techniques to prove that companies should continue production until the change in cost equals the change in revenue. In economics, this relationship is the familiar equation $marginal\ cost = marginal\ revenue$.
Finally, cadets demonstrated the law of diminishing returns to labor with their single variable calculus skills.

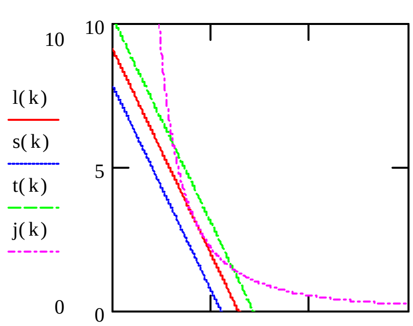
In the second requirement, we gave the cadets a table of discrete data representing production quantities given specific numbers of laborers **and** plants because both of these parameters are allowed to vary in the long run economic model. Using this table, the cadets had to plot the discrete data and develop a discrete dynamical system to model the amount of output based on the inputs. They were also asked to discuss some pros and cons of doing a discrete analysis versus a continuous analysis. Here is an example of one discrete plot:



Half of the cadets were told to hold the value of plants constant and make predictions about production for different numbers of laborers. From the given data, these cadets derived the discrete dynamical system $a(n+1) = 1.2a(n)$. The other half of our cadet groups were instructed to hold the number of laborers constant and examine the effect on production of varying plants. Fixing the number of laborers led to a discrete dynamical system $a(n+1) = a(n) + 408$. Some cadets solved their discrete dynamical system using an analytical solution. Others decided to iterate the recursive sequence in order to predict quantities of production. One of the best aspects of this part of the project is that it led to more than one approach to problem solving.

In Requirement III, cadets were again required to analyze long run production decisions for a firm. In this case, cadets used a continuous analysis rather than a discrete analysis. The cadets were asked to find all extrema for given production functions and classify the extrema using the Second Derivative Test. Then, given a cost equation, they had to explain the mathematics behind the economics terms isoquant and isocost. These two terms refer to level curves for the production and cost curves, respectively.

We gave the cadets a specific Cobb Douglas Model in Requirement IV and asked them to use the Method of LaGrange to determine the optimal budget the company should strive to meet. Instead of relying on the traditional system of equations method for the Lagrange Multipliers, we had the cadets plot their constraint equation and three selected level curves of their objective function.



After constructing the following plot, the cadets had to use the concept of parallel gradient vectors to select the correct maximum output level curve. The idea of using mathematical software to plot the objective function level curves and constraint equation to further understand Lagrange Multipliers was originally developed by MAJ Mike Johnson and has been used successfully for several years in our department. The final aspect of this requirement was to describe what happens to an isoquant as technology advances, or in economics terms, in the very long run. The cadets had to determine that the effects of technology over many years would cause the isoquant curves to shift, meaning that fewer people and plants could produce the same output.

The project benefited both cadets **and** instructors. The hard work cadets put into solving the problem proved successful in deepening their understanding and appreciation of the power of mathematics to solve real world problems. The project was also successful in furthering the cadets' understanding of the concepts they were learning in their economics course. Some of the concepts and formulae encountered now made more sense when their mathematical development was explored. The math and economics instructors learned a tremendous amount about the mathematical foundations for several major concepts in economics. And finally, this project was a great "carry – through" problem that beautifully tied together major concepts from three semesters of calculus and one semester of economics.

For someone thinking of assigning a similar ILAP, there are a few recommendations we would make. Cadets spent a lot of time on this project; some teams of two spent up to 30 man-hours on the project. This was obviously much more than we allotted. A major reason for the time sink was their weakness in technology skills. Cadets spent a huge amount of time learning our math software program. Recently, we incorporated into MA205 a series of problems that had to be solved by computer so that when project time came, the cadets were more prepared. It worked! The bottom line is that you should emphasize the needed technology skills in class prior to the project so that more student time can be focused on the concepts of the project.

One of the hardest parts in the development of this project was trying to determine topics in the calculus courses that could be applied to economics. Additionally, as with any ILAP, coordinating the schedules of the instructors from two different departments caused the planning and writing process to last several weeks. The lesson learned was to start early!

An Interdisciplinary Project at USMA
LTC Steve Horton

The United States Military Academy (USMA) offers a four-year experience that has significant academic, physical, and military components. The academic program culminates in the award of the Bachelor of Science degree for each graduate. The Military Academy's general educational goal is clearly stated in its Concept for Intellectual Development: "To enable its graduates to anticipate and to respond effectively to the uncertainties of a changing technological, social, political, and economic world."

Mathematics, Science, and Technology at USMA

Under the general education goal, the Dean of the Academic Board has established a set of nine Academic Program goals. To attain these goals, cadets are required to take a large set of "core" courses; 16 in the humanities and social sciences and 15 in mathematics, science, and engineering. The latter consists of 4 mathematics course, 2 physics courses, 2 chemistry courses, 1 computer science course, 1 terrain analysis course, and 5 engineering science/design courses. All of these, except the engineering courses, appear in the first four semesters in the typical cadet's academic program of study.

One of the Academic Program goals calls for cadets to "understand and apply the mathematical, physical, and computer sciences to reason scientifically, solve quantitative problems, and use technology." We shall refer to this as the *mathematics, science, and technology goal*.

MA206

The four core mathematics courses at USMA, in the order they appear, are Discrete Dynamical Systems, Calculus I, Calculus II, and Probability and Statistics. MA206, Probability and Statistics, is a 40 semester-hour calculus based introduction to probability and statistics. The course covers descriptive statistics, classical probability, point and interval estimation, hypothesis testing, and an introduction to linear regression. Due primarily to its place at the end of the core mathematics program, MA206 was designated to include a student project that integrated concepts from several mathematics, science, and engineering courses at USMA.

The Project

In the spring term of Academic Year 98-99, the cadets were asked to do a project that relates to the mathematics, science, and technology goal that is outlined above. This project, called the "Math, Science, and Technology Interdisciplinary Lively Application Project", or MST ILAP, was written in collaboration with the Department of Civil and Mechanical Engineering. The project placed the cadets in a scenario as engineer lieutenants deployed to Thailand. The situation required them to investigate a newly reconstructed dam and associated water levels. The project was quite open-ended and required cadets to call upon skills they learned in MA206, as well as skills from previous math and physics courses. The project can be downloaded by going to our web site at http://www.dean.usma.edu/math/CORE/ma206/ and following the "Projects" link.

The Results

By several different measures, the MST ILAP was successful. First and foremost, the cadets demonstrated that they "understood and could apply the mathematical, physical, and computer sciences to reason scientifically, solve quantitative problems, and use technology" as called for by the mathematics, science, and technology goal. Due to the subjective nature of the assessment process, the evidence supporting this conclusion is largely anecdotal, but we have lots of technically correct and well-written reports that demonstrate achievement of the MST goal.

The MST ILAP was also successful from the standpoint that it gave the cadets an opportunity to exercise some of the skills they learned in MA206. Many cadets who didn't completely grasp important course concepts such as normality testing, confidence interval construction, and hypothesis testing were able to learn by doing on the project. Finally, the cadets enjoyed it! Although the open-ended, unstructured nature of the project caused some anxiety, most cadets overcame this and were able to produce an excellent product. We intend to continue the MST ILAP idea in future semesters. Watch our web site for details!

Corneal Topography

Dr. James Rolf, USMA

Introduction

Ophthalmologists would like to have an accurate representation of the anterior corneal surface. This representation is used to design contact lenses, eyeglasses, and to reshape the cornea via surgical procedures to improve eyesight. Since we are unable to measure the cornea with physical objects, ophthalmologists need a *remote sensing* technique to produce the desired representation of the corneal surface. The field of study related to this endeavor is known as corneal topography. In the real wold, corneal topography is used to approximate the shape of the cornea and is useful to ophthalmologists in surgery to correct poor eyesight as well as to designers of individual contact lenses and eyeglasses.

I explored this topic of corneal topography via a project with a highly motivated first year student in my second semester calculus[1] in lieu of another required project. I gave the following information, along with a little background material, to my student.

Student Project

Our goal in this project is to develop such a technique numerically explore the accuracy of our technique.

Corneal Topography

Current corneal topography techniques project a grid pattern onto the cornea and then photograph the reflection of this grid pattern. This produces a distortion of the original grid pattern.

The mathematical problem is an inverse problem—that is we start at the end and try to work our way back to the beginning. In this case, we start with the distorted pattern and ask ourselves the question, "What surface created this distorted pattern?" So we are interested in gaining more understanding of the mapping f^{-1}.

The corneal surface is a two dimensional object (because it can theoretically be described by a two variable function $p(x,y)$.

We will simplify the problem by examining a one-dimensional slice of the corneal surface that can be described by a one variable function $p(x)$

One Dimensional Cornea

We presume that the cornea is a convex shape described by some unknown function $\mathbf{P}(x)$. So each point on the corneal surface can be described by the vector $\mathbf{P}=(p_1(x),p_2(x))$. Our known grid pattern is the single curve $\mathbf{S}(x)$. We will describe each point on the grid pattern by the vector $\mathbf{S}=(s_1(x),s_2(x))$.

Rays of light emanate from each point, **S**, on the grid pattern. We will assume that only one ray of light bounces off of the cornea at the unknown point, **P**, reflects around a normal vector, **N**, and travels through the camera lens to the film plane. So we know the starting point, **S**, and the final point, **F**, and would like to determine the unknown reflection point, **P**.

We should note that the location of the reflection point, **P**, is dependent on the source of light, **S**, so $\mathbf{P}=\mathbf{P}(\mathbf{S})$. Thus the resulting film data point is also a function of **S** since $\mathbf{F}=\mathbf{F}(\mathbf{P})=\mathbf{F}(\mathbf{P}(\mathbf{S}))$. Since we know **F**, we can construct a unit vector **U** that points in the same direction as **P** and can thus consider our reflection point as the vector

$$\mathbf{P} = r\mathbf{U}$$

Since **P** is a function of **S**, both r and **U** are functions of **S**, as is the normal vector **N**. But since $\mathbf{S}=\mathbf{S}(x)$, we also observe that r, **P**, **F**, **N**, and **U** are all functions of x.

[1] MA104 is usually taken in the spring semester of plebe year (first year). This course includes some integral calculus and some differential equations.

Thus one important observation about all of this is that we can describe this system as a function of one variable.

A second important observation is that **U** and **S** are known quantities. Since we know the starting point of one ray of light (the source grid **S**) and we have the distorted image of this ray of light (on the film plane at **F**), we also know the unit vector **U.** Since we are trying to find the unknown point **P**, what we really need to find is the distance from the origin to **P**, which is r. So our problem boils down to determining r.

Now we would like to understand the fundamental differential equation that governs this process. We will do this by constructing this equation.

Part A: The Fundamental Differential Equation

1. Here's some things to consider as you do this:
   a. Re-write **P** as $\mathbf{P} = r\mathbf{U}$.
   b. Construct a normal to the corneal surface, **N**, as a linear combination of **S**, **P,** and **U**.
   c. Use the fact that a tangent vector, $\frac{d\mathbf{P}}{dx}$, and the normal, **N**, are orthogonal.
   d. You will also need the fact that since **U** is a *unit* vector, $\mathbf{U} \cdot \mathbf{U} = 1$. So what does this imply about $\frac{d\mathbf{U}}{dx} \cdot \mathbf{U}$?
   e. Form the d.e. $\frac{dr}{dx} = stuff$

Part B: Solving the Fundamental Differential Equation Numerically

1. Assume that the source curve is $\mathbf{S}(x) = \left(20x - 10, 4(x - \tfrac{1}{2})^2\right)$ where $x \in [0,1]$. The camera lens is at $\mathbf{S}\left(\tfrac{1}{2}\right)$. The film plane lies on line $y = -a$. Assume $a = \frac{1}{2}$. The initial condition is $\mathbf{P}(\tfrac{1}{2}) = (0,10)$.
2. Use forward Euler's method to solve this differential equation, given supplied data for **F** and **S**.
3. Use a fourth order Runge-Kutta method to solve this differential equation, given supplied data.
4. Use the implicit backward Euler's method to solve this differential equation, given supplied data

Part C: Interpretation and Analysis of Results

1. What recommendations do you have regarding our *theoretical* model of the corneal surface? What thoughts do you have about any assumptions we made (or didn't make)?

2. Compare and contrast the theoretical and actual accuracy for each of the above methods. Which one do you recommend for use in this problem? You should consider accuracy, stability, and computational cost, along with requirements of the ophthalmologist who will use your algorithm in making surgical decisions.

3. Make any relevant comments about any assumptions we made in the implementation phase of this project.

How the project worked.

The student I gave this project to was a very motivated first year student who had demonstrated mastery of the material that we were studying in class and had a strong mathematical background. Additionally, he was a willing worker. In fact, he had requested enough homework from me so that he could work three weeks ahead on the assignments! I believed that he would be the kind of person who would welcome and be motivated by mathematical challenges. So I talked to him to see if he was willing to do an alternative project that would challenge him more than the planned project. I also told him that I would slant the project heavily towards numerical issues since I realized that he had strong interests in computational computer science (He was considering majoring in computer science at the time. I am now happy to report that he is planning on a double major in mathematics and computer science). He was eager to do so.

This project has two parts. The first part is a theoretical component in which the student derives a differential equation. The second part of the project involved implementing code to numerically integrate the derived differential equation. I required two major in progress reports. The first to insure that he was headed in the right direction for the derivation of the differential equation and another to deal with any numerical/programming issues that might come up.

The first part of the project, developing the governing differential equation, required some basic knowledge of vectors and dot products. This material is something that we had not covered in the first two semesters here at West Point. But since the student had a strong background and was highly motivated, he was able to figure out (with guidance) the information he was not familiar with. The derivation itself required several hints on my part and strong symbol manipulation skills on the part of the student. But, he eventually got through this and came up with

$$\frac{dr}{dx} = \frac{r\,\frac{d\mathbf{U}}{dx}\cdot\mathbf{S}}{r+|\mathbf{S}-r\mathbf{U}|-\mathbf{U}\cdot\mathbf{S}}.$$

The second part of the project involved numerical computation. Since the cadet was very interested in computing, it was fairly easy to keep him engaged on this front. I asked him to numerically solve the differential equation three different ways—using the explicit forward Euler's method, an implicit backward Euler's method and an explicit Runge-Kutta method

Since we had discussed Euler's method in class, the cadet quickly implemented forward Euler's method,

$$r_{n+1} = r_n + Dr(r_n, \mathbf{U}_n, \mathbf{S}_n),$$

where

$$Dr(r,\mathbf{U},\mathbf{S}) = \frac{dr}{dx} = \frac{r\,\frac{d\mathbf{U}}{dx}\cdot\mathbf{S}}{r+|\mathbf{S}-r\mathbf{U}|-\mathbf{U}\cdot\mathbf{S}}.$$

Since the student was only given data for F at discrete points and for **S** at discrete points, he needed to calculate the unit vector **U** for use in these numerical schemes. He also needed to approximate $\frac{d\mathbf{U}}{dx}$ for use in the differential equation. But since **U** is only known at discrete points, the cadet had to "discover" this need to approximate $\frac{d\mathbf{U}}{dx}$ and figure out a way to do so. And in fact he did this using a first order approximation by finding the slope of the secant line between two points.

The student properly realized the forward Euler's method gave increasingly poor results as he integrated to the outer edges of the simulated cornea. This is not desirable for an ophthalmologist considering surgery on the cornea!

The student then implemented a fourth order Runge-Kutta scheme. This required some outside reading by the cadet, but given his computing interests, he quickly discovered the scheme

$$r_{n+1} = r_n + \frac{\Delta x}{6}\left(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}\right)$$

where

$$k_{n1} = Dr(r_n, \mathbf{U}_n, \mathbf{S}_n)$$

$$k_{n2} = Dr(r_n + \tfrac{\Delta x}{2}k_{n1}, \mathbf{U}_{n+\frac{1}{2}}, \mathbf{S}_{n+\frac{1}{2}})$$

$$k_{n3} = Dr(r_n + \tfrac{\Delta x}{2}k_{n2}, \mathbf{U}_{n+\frac{1}{2}}, \mathbf{S}_{n+\frac{1}{2}})$$

$$k_{n4} = Dr(r_n + \tfrac{\Delta x}{2}k_{n3}, \mathbf{U}_{n+\frac{1}{2}}, \mathbf{S}_{n+\frac{1}{2}})$$

This, of course, gave much better results when compared to the forward Euler scheme. The student also noted the increased computational cost when implementing this scheme.

Finally, the student implemented an implicit backward Euler's method. This scheme can be described as

$$r_{n+1} = r_n + \Delta x Dr(r_{n+1}, \mathbf{U}_{n+1}, \mathbf{S}_{n+1}).$$

The important thing to note is that this is an *implicit* method that requires using a numerical technique, such as Newton's method or a secant method, to solve for r_{n+1}. The student had difficulty implementing the secant method required for this technique, but was finally able to compare the higher accuracy of Runge-Kutta with both the implicit and explicit schemes.

The student then discovered that this technique did not have as much accuracy as Runge-Kutta. But I do not think he properly appreciated why it is important to consider the more stable implicit methods when numerically integrating. In hindsight, I realize that I did not motivate the necessity of using mo re stable schemes. So next time I need to provide data that could have resulted in unstable results with explicit methods.

Reflections on this project.

I am very happy with how this project turned out for this student. As it is written, the project worked well for a student with strong symbol manipulation skills, some background in vector calculus, and a strong interest in computing issues. But other variations of this project could be adapted to different courses with different kinds of students.

For example, this project could be adapted to a vector calculus course by requiring more exploratory activities involving construction of normal vectors, reflected vectors, tangent planes, etc. The computational requirements could also be avoided by using "black box" approaches with Maple, Mathematica, MathCad, MatLab, and/or other computer algebra systems.

This project could also be adapted in a myriad of ways for a numerical analysis course by decreasing the emphasis on the derivation of the differential equation and exploring different ways to approximate $\frac{d\mathbf{U}}{dx}$ via polynomial interpolation, least squares techniques, spline techniques, and/or wavelet techniques. Noise could also be introduced to the data to provide a more realistic situation. Additionally, one could also examine the two dimensional structure of the differential equation and explore techniques for solving the resulting system of differential equations.

Please feel free to contact me at rolf@usma.edu or at (914) 938-8135 if you have further questions.

A Project that Worked(?)

MAJ. Philip Beaver
USMA

I have yet to write a project that was not intended to achieve dozens of lofty goals for student learning, student discovery, student enrichment, student self-actualization, and student nirvana in terms of the Mean Value Theorem or some other topic *du jour.* I have frequently attained student confusion and not much else, which only compels me to greater lengths to intricately compose the perfect project on my next attempt.

There are several mines I have blissfully tripped as I have strolled down the project-writing path. These include wrapping accessible mathematics in an inaccessible application, wrapping an application around inaccessible mathematics, wrapping a ludicrous application around meaningless mathematics, and perhaps every other possible type of wrapping in the name of "interdisciplinary applications."

I have assigned one project which achieved a few goals (none of which I had intended) and actually resulted in positive feedback from across the student spectrum. In retrospect, I believe I have even figured out why it was successful, and I would like to share the wisdom only attainable through 20/20 hindsight. I called it "The Rattlesnake Problem."

The setting is MA103, Discrete Dynamical Systems and Introduction to Calculus, halfway through the first semester of freshman year. We had already touched on several population models in our development of linear, nonhomogeneous DDSs (deer growing unchecked, using hunting quotas to keep the population stable, amoebas growing in a petri dish with the occasional amoeba-eating shark swimming by, and even a few silly applications). The students had attained a reasonable comfort level in terms of modeling simple populations and analyzing their stability in terms of the solutions to the DDSs. It was at this point that I assigned the Rattlesnake Problem.

An article had appeared in West Point's *Pointer View* newspaper the previous summer, entitled "Rattlesnakes on Threatened Species List." James Beemer, the fish and wildlife biologist for West Point, explained how the West Point timber rattlesnake population was declining, and was perhaps soon to join several extinct populations from the northeastern states. Along with a plethora of general information about rattlesnakes, the article stated several factoids about their population and reproduction. These included average life span, mating age, mating frequency, litter size, probability of survival in the first year, and the lack of correlation between age and number of rattles. One crucial piece of information, the average life span of snakes in the West Point population, was missing.

The assignment was simply to model the population based on the article, determine the long-term behavior, and report to Mr. Beemer on the population's stability, all for a 50-point consulting fee. One end of the response spectrum included an exact regurgitation of the model we developed for the amoebas with hourly shark attacks. However, the other end included thoughtful and detailed analysis which identified missing information, made assumptions, considered environmental variables, proposed further studies, offered solutions in terms of unknown parameters, and even proposed protective measures (such as, canceling Cadet Basic Training). The reaction was unexpected. The majority of the students claimed they were "challenged," "intrigued," and that they had actually learned something, in addition to gaining confidence, appreciation, and nirvana.

So why did this project work? First, the application was accessible. The only background required came from a simple newspaper article. Second, with no clear solution, the students were able to rise to their own individual levels. Most of them recognized the problem as being "open-ended" and they felt free to explore. Finally, the problem included most of the material we had used to that point in the course, while requiring that the best solutions extend slightly beyond that material. The students did this within their own capabilities.

The clear lack of an exact solution to a comprehensive problem, coupled with an accessible application, seem to be the key ingredients in this project's success. Perhaps the simplicity forced on me by the 50-point limit kept me from repeating my past mistakes. Maybe my students conspired to give "positive responses" as a cruel joke on their instructor. Whatever the reasons I will always think of "The Rattlesnake Problem" as one that worked.

Projects in Math Courses: Can a Good Thing Get Even Better?

Dr. Ethan Berkove and Dr. Richard Marchand

USMA

During the last couple of years we've developed a half dozen projects in both the core courses and electives. We are firm believers in projects for a variety of reasons. For example, they give students practical experience in many mathematical applications, motivate new topics, provide interdisciplinary relationships, and incorporate modeling and technology, ..., and the list goes on. They also give students important practice in technical writing. After all, what good is a result if no one can understand what you've done? However, writing a good project shares some of the same challenges of speaking to someone for whom English is a (distant) second language; you know what you're asking for and try to speak simply; but often when you look at the final product, it appears something was lost in translation. We'd like to discuss a couple of factors that we believe are important to help make a project a success for both the teacher and the student.

A project assignment is a perfect way to answer the question "What is the mathematics I'm studying good for?" In some respects this is an easy question, for mathematics is ubiquitous in science, and almost all undergraduate mathematics arises in some application somewhere. A common difficulty is that many of these applications are complicated or require significant non-mathematical background. In an effort to build interesting projects, it's perhaps too easy to include significant components of other fields that actually obscure the mathematics, rather than reinforce or develop it. In a number of projects we've assigned here, **instructors** have walked out of project briefings confused or with a partial understanding of what's been asked. It's not hard to imagine, then, that the same project scenarios have students pulling out their remaining hair trying to figure out what to apply and how to apply it.

We see a number of potential ways to avoid this dilemma. If the application is not an issue, one option is to avoid incorporating any major concepts from outside of mathematics. A project in this form would build a scenario around a mathematical skill. The problem might be somewhat contrived, but should still develop problem solving skills.

Examples of this type of project might be the analysis of a flight path (arc length, curvature, self-intersection, etc.) given by a system of parametric equations, or the solution to a differential equation. Another option is to incorporate another field, but include only elementary concepts. Here, a project goal might be to gently introduce students to a new topic from a related field, like resonance or free body diagrams from physics.

Sometimes, however, you come across a really great idea from another field that simply begs to be turned into a project. In these cases we firmly believe you should grab the ball and run with it. But certainly note that the students will be doing double duty, trying to master both the subject matter of the new field as well as the mathematics. This extra time needs to be taken into account, and you'll have to become enough of an expert to potentially provide guidance in two fields (or more). Plan to add extra class time devoted to the discussion of background examples and material. Special handouts might also be appropriate. In any case, this is probably a good time to talk with colleagues in the partner discipline and consider their comments and suggestions. Another idea is to require students to maintain a journal, recording their time on task and difficulties encountered. This information can be valuable for planning future projects.

Another factor to consider in project assignments is the exposition. It's an aspect easily taken for granted. We, the teachers, want to see a clear, well-explained, well-organized proofread report. When a finished product approaches 25-30 pages, this is a major undertaking. It is notoriously difficult to mesh text, mathematics, and diagrams – students work hours on the finishing stages. In fact, this may take so long that it directs student attention away from the mathematics, and the quality of the report suffers. An obvious solution here is to allow more time for the write-up and clearly inform the students of your higher expectations. Here's another idea we have been contemplating: Ask for a typed executive summary, graded for style, presentation, and content. Other supporting works, like graphs, charts, calculations, justification, and so on, can be handwritten. The project grade would then be split between an analysis of the mathematics and the quality of the professional component. This grade would reflect both mathematical ability and communication skills, while hopefully reducing the students' time commitment. This option might be particularly appropriate for courses requiring more than one project.

We believe that projects are one of the strengths of the mathematics curriculum at West Point. We've repeatedly heard students list these projects as the most useful part of the course. This is evidenced by the quality of some of their reports. However, their biggest complaints have focused on the length of the projects and the material they felt they were required to learn from non-mathematical topics. While adversity builds character, excessive loads **can** break backs. We have cited some ideas we think may help relieve these problems, and welcome any reader comments.

A Project for Engineering Mathematics

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The course Engineering Mathematics is composed of three major topic areas: vector calculus, solutions to ordinary differential equations using Laplace transforms and introduction to partial differential equations. Throughout the course, we have traditionally given two projects: one for vector calculus and the other either an ordinary or partial differential equation problem. During the first semester, we tried to develop a project that would span both the ordinary and the partial differential equation block.

The idea was to model the effects of an earthquake on a two or three story building. The first part of the project modeled the vertical shock to a building. The students developed a system of ordinary differential equations that modeled this displacement using a spring-mass system as the basis for their model. The walls of the buildings were used as the coefficients of the springs, and mass of each floor was given. Ultimately, the students looked at the effects on both a two and three story building. Additionally, the forcing function, which drove the system, was an impulse or a Heaviside function to simulate the effects of an earthquake.

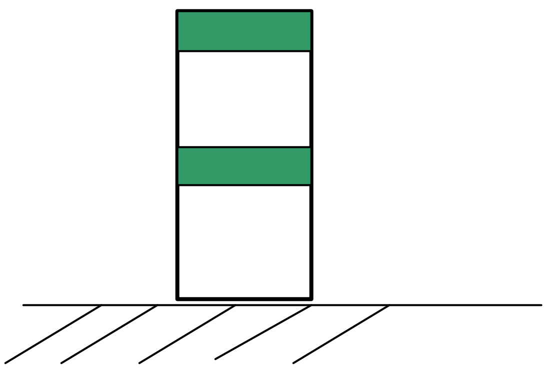
Here is how the first requirement was presented:

1. Introduction

An examination of the application of differential equations to mechanical systems is valuable for many reasons. As an engineer, you can gain insights to the physical behavior of specific systems. Additionally, familiar models, such as the spring-mass system, are used to model the behavior of more practical systems like a building. We will first examine a simple model of a two-story building, then add conditions which make the system more realistic. Specifically, we will add a forcing function simulating the rolling motion of an earthquake.

2. (100 points) The Model

Consider the two-story building below:



We can model this as a spring mass system. Consider the first floor as mass1 and the second floor as mass2. The walls of each floor act as springs supporting the floors, consider these spring constants k1 and k2. Solve your system of Ordinary

Differential Equations if m1=3000slugs, m2=3000slugs, k1=6000 lbs/ft. and k2=6000 lbs/ft. Now if a force, $F = \sin(3t)$ lbs, is suddenly applied upward on the first floor when the system is at equilibrium, find the displacements of the first and second floors as functions of time. Damage occurs to our structure when any floor displaces more than 14 inches. Now apply forces of $F_1 = 1000\sin(3t)$ lbs and $F_2 = 3000\sin(3t)$ lbs upward to the first floor. Does our system eventually fail? If so, when? Analyze this system by altering the mass of each floor and the spring constant of the walls to make a more stable building. Can you make any general recommendations for future building in earthquake zones?

The cadets did very well with this requirement. They had little difficulty developing the systems of equations and did a very nice job analyzing the response the building had to the forcing functions. Additionally, the cadets used MathCad extensively in the plotting of the solutions.

The second requirement was to look at the horizontal displacement of the building due to the shear force at the base of the building.

3. (150 points) The More Complex Model

Previously we modeled our building being subjected to a force in the vertical direction. A wave type of function simulating the rolling ground during an earthquake. Earthquakes also produce a shearing type of motion.

We will develop the equation governing the transverse vibration of a straight beam without damping. The model used is a fourth order Partial Differential equation:

$$m\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial y^2}\left[EI\frac{\partial^2 u}{\partial y^2}\right] = 0;\quad 0 \le y \le L\ ,\ t \ge 0$$

Where: $u(y,t)$ = displacement function, function of time and position

E = modulus of elasticity of the beam material

I = Moment of inertial of the beam cross section

M = mass per unit length of the beam

Using method of separation of variables, find the possible product solutions to this partial differential equation. Consider all three cases. Hint use l^4 as the separation constant.

Next, consider only the case where the separation constant is less than zero, we have an ordinary differential equation of the form:

$$Y^{(iv)} + \frac{l^4}{b^4}Y = 0$$

where l^2 is the natural frequency in radians per second.

Our support conditions yield the following boundary conditions:

At the fixed end, x=0, the displacement and slope are zero.

$$u(0) = 0 \ , \ u'(0) = 0$$

At the free end, y=L, the bending moment and the shear are both zero.

$$u''(L) = 0 \ \ , \ u'''(L) = 0$$

Find the eigenvalues and eigenfunction that will satisfy the fourth order boundary value problem.

Note: This model assumes the base to be fixed. A more realistic model would assume the base to be on a roller. This more realistic model can only be solved numerically, so our project represents some sort of medium level model, which we can solve in this course.

Even this "easier" model cannot be solved completely analytically. You will see this fact as you solve for the eigenvalues and eigenfunctions. In your calculations, you may find the following relationships helpful:

$$\frac{l_n}{b} L = 1.8751, \ 4.6941, \ 7.8548, \ 10.996$$

$$for \ \ n = 1,2,3,4$$

$$\frac{l_n}{b} L = \frac{(2n-1)p}{2}$$

$$for \ \ n \geq 5$$

The cadets did well in separating the variables and applying solution techniques they had learned earlier in the course. However, the problem required some research from the cadets. The additional information requiring them to fully solve this problem was available in our own Mathematical Library but many cadets did not take advantage of this resource.

This project was well received and the cadets came away from it better able to apply concepts developed within the course. It was also worked well breaking the project into two different submissions.

The major force behind developing this project was Dr. Rich Marchand. His insights and research led to a much-improved project from the first draft. Additionally, numerous faculty members from the Civil and Mechanical Engineering Department were involved.

Derivations and Visualizations for the Hydrogen Atom

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The course in the study of partial differential equations at West Point is usually attended by a combination of mathematics and physics students. Seeking a common topic of interest, faculty from these departments developed a project based upon the usual physics students' investigation of the atomic structure of the hydrogen atom. Mathematically, this involves using separation of variables to solve (or, more accurately, investigate), the Schroedinger equation for the magnitude of the electron's probability density. We use interdisciplinary projects in this course to demonstrate the applicability of the material to Math majors for the applications they will wrestle with in their majors courses.

Most quantum mechanics texts treat the hydrogen atom through separation of variables, but all skip many steps and ask the reader to take many things on faith, such as the application of boundary conditions and the discrete nature and values of some of the quantum numbers. This project requires students to work through the entire analysis using maple to explore and fill in the gaps at the places where we know students traditionally have trouble (where the associated Legendre and Laguerre equations become involved). The objectives of this effort were to develop students' ability to use the method, to demonstrate the utility of mathematics, and to coordinate math and physics curriculum.

Analysis:

The course introduces the students to the use of the method to solve PDE in two, and then three variables in the rectangular coordinate system. Work then progresses to multivariable PDE in other coordinate systems. By the time the students encounter the project herein described, they are somewhat familiar with non-rectangular coordinate systems, but the application of the method of separation of variables to this equation is nonetheless very challenging. Careful coordination between math and physics used and in the nature of information sought and conclusions drawn allows the students to negotiate the solution process and appreciate the effort involved, the power of mathematics, and the satisfaction of confirming independent theory and empirical observation. They are lead to the expression for the atom's energy levels, which reassuringly agrees with the Bohr model of atomic structure. A central objective of the mathematics project is to take the students through the process of investigation to discover that the quantum numbers are in fact directly related to the separation constants (eigenvalues) in the method of separation of variables and that their values are determined by the solutions of the ODE (eigenproblems).

Visualization

A key part of this project is visualization of the solution. Typical treatments provide the student with static pictures of either unseen or underived eigenfunctions. We demonstrate how to use Maple to let students visualize and experiment in order to get a better feel for how the math translates into physics. The capabilities of a computer algebra system give us the ability to produce stil more verification of graphical information presented elsewhere. In the study of atomic structure in chemistry or physics, the student often encounters figures depicting the hydrogen atom's electron probability density for various sets of quantum numbers. In one chemistry text [1] when presenting the electronic structures of atoms, the authors show "photographs of mechanical models" of electron cloud diagrams to help explain the structure of the electron's probability density. In this text the authors confess that, "The shapes of orbitals… are **pictorial representations** of the mathematical solutions of the Schroedinger equation. **They do not represent reality**; the shapes are not pictures of electric charges or of matter."

While accurate, we suggest that the underlined phrase has been overtaken by the graphical power of today's computer algebra systems. We can now help students to demonstrate to themselves that these 'pictorial representations' are indeed the real depictions of the solution to the mathematical model of the hydrogen atom's structure. Using the solutions of the separated ODE's, we can generate isoprobability density surfaces. For various values of the quantum numbers, we can (with appropriate scaling) produce three-dimensional surfaces, which recapture the figures seen in many chemistry and physics texts. The real benefit of these depictions comes from students being able to generate these from analytic expressions, and from allowing them to interactively rotate the images and manipulate them by choosing different values of the iso-surface to graph.

After following the arduous trail left by those who first successfully applied the method of separation of variables to the Schroedinger equation, the ability to use the results to generate graphics identical to those seen previously is a great positive reinforcement for students. We find that this project helps motivate the material, previews a difficult subject for the physics majors, and helps translate between mathematics and physics in a way that usually is not done in either course alone.

References:

[1] Brescia, Arents, Meislich, and Turk, *Fundamentals of Chemistry,* Academic Press, New York, p. 150 (1970).

Service Academy Student Mathematics Conference: An Overview

LT. Marc D. Lucas, US Naval Academy

The midshipmen at the 9th Annual Service Academy Student Mathematics Conference had the following presentations relating to interdisciplinary fields:

"Serrin's Vortex Revisited," MIDN 1/c James Coleman. Investigated boundary conditions for describing a tornado for Navier-Stokes equations in fluid dynamics.

"Kinematic Models for Hurricanes," MIDN 3/c Whittemore. Presented a simple model for hurricanes and their circulation within, assuming that the hurricane was a Rankine vortex divided into two circular regions with steady flow of ideal fluid and a vortex jump between the boundary of the vortex and outer region to simplify the dynamic model.

"A Statistical Method for Analysis of Error," MIDN 1/c Joshua Wood. Trident research project on a statistical method for analyzing the error on predictions made through the process of time-delay-embedding of chaotic time series could be applied to nearly any set of chaotic data and its forecast from motion to daily temperatures to stock prices.

"Tiling Infinitely in Three Flavors," MIDN 1/c Kate Oliver. Investigated periodicity, symmetry relations, nonperiodic and aperiodic tiling. The aperiodic tiling, in particular, has applications in the natural work conceptualized by the new science of quasi crystals.

"Determining Buoy Pattern and Ownship's Position from Bearning Information," MIDN 1/c Beer. Research directly applicable to a submarine determining its location in a mine or sonobuoy field of an unknown pattern, like a trumped up game of "Marco Polo."

"The Symmetric Level Index System of Computer Arithmetic," MIDN 1/c Lincoln. An investigation of an alternative to floating point binary numbers and their limitations and memory storage requirements.

"A Solution to Ulam's Problem with Error-Correcting Codes," MIDN 1/c Montague. Examined encryption and different associated efficiencies from the perspective of a user, reader and third party.

It was found that the conference was enriched, motivating and a professionally rewarding experience. The students' ability to develop mathematical concepts, think critically, write technically and make professional presentation was enhanced.