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OF THE FEDERAL SERVICE ACADEMIES

RESEARCH & TEACHING: HOW DO THEY REINFORCE EACH OTHER?

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EDITOR'S NOTES

Welcome to Academic Year 2004-2005! I trust that your summer activities have adequately recharged your professorial batteries in preparation for the coming semester.

This installment of *MM* explores the relationship between research & teaching – an ongoing balancing act that is sure to elicit commentary from anyone who's tried to manage it. Seven of our colleagues have contributed articles that offer thought-provoking insights into this aspect of our professional lives.

We begin with a piece in which USMA's Lieutenant Colonel Mike Huber argues for a very natural merging of the complementary tasks of research & teaching – the result of which he dubs *reaching*.

Building upon this notion, USMA's Colonel Darrall Henderson and Lieutenant Colonel Tyge Rugenstein present West Point's current structure for harnessing faculty research talents and providing research opportunities to our cadets.

Next, USMA's Dr. Heather Dye explains how she translates her own research habits into helping her students develop their own problem-solving skills.

In a similar vein, USAFA's Dr. Michelle Ghrist recounts how her puzzle-solving mindset endows her research with a sense of 'play' rather than 'work' and how she tries to share that point of view with her students.

After that, USMA's Professor Brian Winkel serves up a sampler of the latest fruit that he's harvested from a recurring research venture that he pursues with cadets.

Finally, USMA's Dr. Lee Zhao reminds us that our own knowledge base – the deeper the better – translates directly into our confidence as instructors and as academic role models for our students.

Although the biannual publication rhythm for *Mathematica Militaris* will continue, I intend to break the recent pattern of "theme" issues. Rather, future Calls for Papers will solicit articles of general interest to the readership. I hope you enjoy the present rendition and that you will become inspired to share your own ideas, techniques, and strategies with your colleagues in future volumes.

Be sure to visit our website for past issues: *http://www.dean.usma.edu/math/pubs/mathmil/*.

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"Reaching" – Combining Research with Teaching

Lieutenant Colonel Mike Huber,
USMA, Department of Mathematical Sciences

One of our primary missions as mathematics professors is to help students grow as life-long learners by formulating intelligent questions and asking them to research answers independently and interactively. Another effort is to conduct research in our chosen fields. Can we combine our missions to teach and conduct valuable research? Can we reach out to students through teaching and research? The answer to both, of course, is Yes! For the past fifteen years, the service academies have participated in the annual Service Academy Student Mathematics Conference (SASMC). In this paper, I would like to give an example of how we, as math professors, can pursue personal research and combine it with our mission to teach.

At USMA, we have a semester-long course, MA491 *Research Seminar in Applied Math Projects*; this is our senior thesis course. The official course description calls for students to "integrate the mathematical concepts and techniques learned in previous courses with the principles developed in the whole USMA Curriculum to solve a current problem of interest to the Academy or other agencies in the Department of the Army. Students may select problems from a list of suitable projects provided by the Department of Mathematics. Students may work individually or in small teams, depending on the nature of the research. The course will culminate with a student presentation and a paper prepared to send to the 'using' agency."

A few years ago, I taught two elective courses: MA385 *Chaos and Fractals*, and MA386 *Introduction to Numerical Analysis*. Numerical Analysis is a required course for math majors, while Chaos & Fractals is simply another elective. One of the basic tools learned early in any numerical analysis course is the use of root-finding techniques. Students quickly learn that the Newton-Raphson (or simply Newton's) Method is a powerful way to solve $f(x)=0$. Convergence is fast and accurate in this iterative approach. Students are always looking for practical applications of the material, so I racked my brain trying to find something new and interesting. I had studied several courses in numerical methods, dynamical systems, chaos, and control theory at graduate school, and I was interested in how fractals were generated.

I noticed that the process of iteration belongs to the intersection of my two electives, and I soon discovered that Newton's Method could be used to generate fractals. I was also curious about how to iterate a sequence of complex numbers to a root of a complex polynomial $f(z)$. Although this is a relatively new field of mathematics, and I had by no means invented this generation technique, I wanted to research different aspects of it. Writing the code to generate fractals iteratively is not particularly difficult, and I soon used it in the classroom to teach cadets. I started dabbling in creating fractals from higher-degree polynomials (one such fractal from a 4^{th}-degree polynomial $f(z)$, where z is a complex number, is shown below in Figure 1).

One student (Cadet Simon Kim) approached me for a topic for his research seminar project. He was in both of my elective classes. Although we had not yet covered the properties of fractals in class, he was intrigued by the self-similar nature of different fractals. He was also curious about rewriting numerical algorithms. I suggested that we try generating fractals from polynomials with complex roots, and a senior thesis was born!

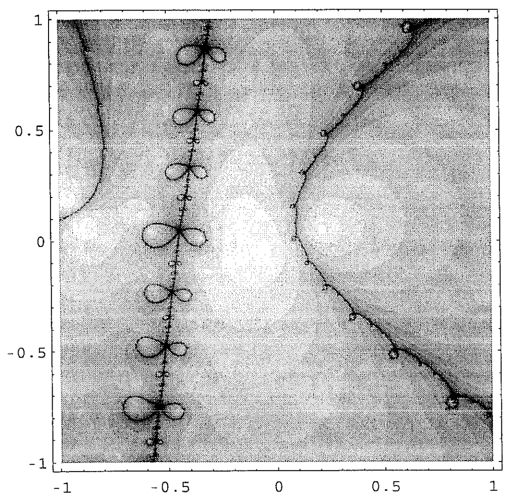

Figure 1. Fractal generated in the complex plane from $f(z)=0$.

In the numerical analysis classroom, we spent several days on different root-finding techniques. In the fractals class, we studied the theorems surrounding the basins of attraction and tried to determine self-similarity of common fractals. Once or twice per week, I would meet with Simon and extend what we covered in class to enhance his project. For a worthwhile MA491 project, as in any research project, I wanted him to dive into a literature study (what has been done?), develop a plan to try something new (what do I want to do?), and then execute the plan. We soon discovered that there were some existing fractal-generating algorithms on the Internet, a few papers about basins of attraction, and lots of information about Newton's Method. As the amount of material available was very large, we soon narrowed the scope of the project to determining the basins of attraction for all fixed points in the plane under a simple domain: $-1 \le x \le 1, -i \le y \le i$. As seen in Figure 1, several "bulbs" appear on the boundaries between basins of attraction. The various light-to-dark shades are indicative of how many iterations are needed to converge to a root (lighter shading implies fewer iterations needed). Our goal was to research the relationships between bulbs and fixed points. This narrow focus was perfect for the six to eight weeks available for "work" once the literature search was finished.

It's been three years since the completion of this first MA491 project. Simon Kim briefed his techniques and results at the 12th SASMC, which was held in Annapolis in 2001. Since then, I have continued adding bits and pieces to the algorithm. During each Fall semester, I advertise what I'm interested in to new students. This past semester, Cadet Brandon Rumbelow jumped on the project and briefed at the 15th SASMC in Monterey and at the Hudson River Undergraduate Mathematics Conference (HRUMC). Brandon and I are now investigating the prediction of boundary points and whether there exist regions containing initial points that do not iterate to a fixed point. The questions asked by other students at both conferences confirmed to me that Brandon's research effort had been worthwhile.

So, what? If you are teaching an electives class that has research interests for you, think about advertising for a senior thesis project. We know the SASMC will occur every year. There are also other opportunities, such as the HRUMC. In addition, since so much interest and discussion was generated at the two conferences, I am working with Brandon to get his work published in a journal dealing with undergraduate research. In the future, I want to encourage a student to present his or her work at a regional MAA meeting.

Next semester, I will be teaching a linear algebra course. Another of my personal research areas is utilizing Singular Value Decomposition (SVD) to study image compression. This is another new and exciting field which shows the practical applications of what we teach in the classroom. I plan to offer a senior thesis proposal linking the SVD and image or data compression, which is currently of interest to several Army agencies. As a bonus, although we are undergraduate institutions, sponsoring a senior thesis allows us to have a "research assistant" while we teach.

Each of our academies has a point of contact for senior thesis seminars. At USMA, we also have an Outreach Officer, who does a great job linking faculty with real-world problems facing the Army. He finds the research passions of our professors and then coordinates with Army or DoD agencies that need analysis. We all show enthusiasm in the courses we *want* to teach, and we all have genuine interest in our chosen research areas. Why not combine those two interests? Called Out-*Reach* for a reason, this is a great opportunity for faculty to combine their research interests with teaching interests. Reaching also gets our students involved outside the classroom. I encourage you to reach out and become a senior thesis advisor next year. Practice reaching versus teaching. You (and your students) will benefit from the experience. ■

Facilitating Outreach and Teaching: The Mathematical Sciences Center at USMA

Colonel Darrall Henderson and Lieutenant Colonel Tyge Rugenstein, USMA, Department of Mathematical Sciences

As an undergraduate institution, the United States Military Academy's primary task is to educate cadets and prepare them for the rigors of their chosen major and the Army. However, the Department of Mathematical Sciences' charter does not end with graduation. The Department continues to develop leaders of character by professionally developing its faculty. The Army and society "loan" the Department exceptional officers and civilian faculty to assist in the education and development of cadets. The Department of Mathematical Sciences at USMA seeks to professionally develop its faculty and return these individuals to the Army and academia as professionally competent field grade officers and scholars ready to tackle the challenges facing tomorrow's Army and the Nation.

But how can the Department blend the academic, professional, and military development of faculty and still maintain a focus on undergraduate education? The Department of Mathematical Sciences at the United States Military Academy is attempting to walk this fine line by implementing several programs that combine outreach with teaching (REACHING – a term introduced by LTC Mike Huber). This article will highlight several initiatives and the structure developed in the Department to facilitate the program's success.

Outreach by the Academy to the Army and Department of Defense (DoD) comes in three forms: student research, faculty research, and ties to the Army Research Laboratory (ARL). Student research is further broken down into Advanced Individual Academic Development (AIAD) and capstone course projects for Math and Operations Research majors (MA491). Faculty research tied to problems in the Army and DoD often initiates or directly supports AIADs and MA491s, many of which stem from the Academy's link to scientists and projects at ARL.

The goal of REACHING at USMA is to tie faculty and student research projects to our mission of educating and training the future leaders of the Army. To facilitate this lofty goal, the Department of Mathematical Sciences maintains the Mathematical Sciences Center (MSC).

The Mathematical Sciences Center

The MSC provides collaborative research opportunities for USMA faculty and cadets that support the Army, Department of Defense, Combatant Commanders, and the USMA community. This outreach provides professional development, enhances instruction for cadets, and promotes USMA as a relevant resource for the Army. The MSC facilitates research by faculty with graduate degrees and experience in Mathematics, Statistics, and Operations Research to address current topics of concern to the Army, DoD, and the Combatant Commands. Additionally, the MSC leverages the Army Research Laboratory to enhance the effects of technology in the Army.

The MSC consists of three pillars: The Center for Data Analysis and Statistics (CDAS), Outreach, and the Army Research Laboratory Liaison. Each of these pillars is unique in its mission, but closely affiliated with the other pillars. This affiliation provides flexibility and synergy in terms of faculty development and research. The MSC is depicted in Figure 1. Note that the ties between the three pillars of the MSC are not formally tied to the three types of outreach that the Department conducts.

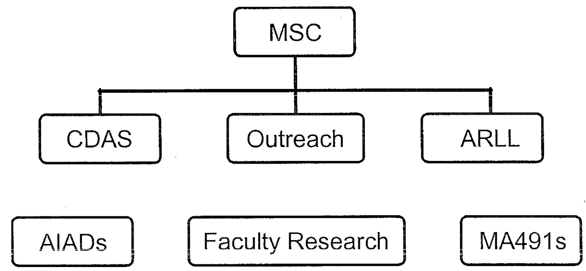

Figure 1. Mathematical Sciences Center Organization

Center for Data Analysis and Statistics

The CDAS provides expertise for problems requiring statistical methods. The Center is geared to help faculty and researchers with statistical questions and to support data analysis projects at USMA and throughout the Army. In some cases, personnel from the CDAS conduct the actual analysis; however, the main focus is to advise and support individuals, units, or agencies in their own data analysis efforts.

The mission of the CDAS is to provide services similar to those available in the statistics department at most major research universities. Many individuals, departments, and agencies at West Point and throughout the Army require statistical consulting. In some cases, the consulting consists of an office call by one member of the CDAS to discuss sampling techniques or design of experiments. The office call is often the "jump start" the client needs to head in the right direction. In other cases, the CDAS is retained by an individual or agency to conduct the actual data analysis and assist with an ongoing study. Both instances involve faculty interaction with current analysis that translates directly into enhancing the classroom experience for cadets. Ideally, efforts by the CDAS will eventually encompass projects and analyses appropriate for cadet participation in terms of classroom projects, MA491 projects, and even AIADs.

Current CDAS initiatives include: work with the orthopedics department at Keller Army Community Hospital on an ACL study; work with the Department of Physical Education at USMA analyzing a remedial physical training program based on data from a recent study of basic training soldiers at Ft. Jackson; and a weapons lubricant study with ARL. Members of the CDAS also assisted two Ph.D. candidates with statistical matters related to their dissertation research.

Outreach – Faculty and Students

In order to bring relevant topics into the classroom and to assist the Army and DoD, the Department encourages its faculty to engage in research inside and outside of the Academy. This outreach not only broadens the experience of the faculty member, but enhances the education of the cadets by bringing lively applications and real world experience back into the classroom. The Department makes no distinction between civilian and military faculty when it comes to outreach. All are afforded the same opportunity to conduct outreach, as long as it is consistent with teaching and other duties. Another benefit of such outreach is that the Army and DoD recognize the Military Academy as having a large faculty talent pool capable of solving current Army problems.

The effort to blend education with real world applications resulted in several outreach initiatives within the Department this year. For example, we successfully placed a number of faculty members in short-term internships with outside agencies such as the Center for Army Analysis, Military Entrance Processing Command, United States Special Operations Command, the Office of Congressional Legislative Liaison, Human Resources Command, the Army Research Laboratory, and the Joint Staff. The goal of each outreach initiative was to develop officers and civilian faculty as well as to provide "value added" to the Army, DoD, and other governmental agencies.

Faculty research often enhances classroom experience by generating a desire to explore and expand beyond the confines of the standard curriculum. These REACHING initiatives by faculty result in classroom projects, MA491 capstone projects, and Advanced Individual Academic Development for cadets. Some examples of senior capstone projects this past academic year included work with the Department of Physical Education at USMA, the Admissions Department at USMA, and the Military Entrance Processing Command in Chicago. In addition to capstone projects, cadets participated in AIADs at the High Performance Computing Center in Hawaii, the Institute for Creative Technologies in California, and Draper Labs in Massachusetts.

Army Research Laboratory Liaison

The ARL Liaison serves as the Army Research Laboratory research coordinator at USMA. The ARL Liaison is the direct link between over 1500 ARL scientists and engineers and the faculty at USMA. The ARL Liaison serves as the coordinator for faculty research between ARL and all departments at USMA. This vital link between state-of-the-art research facilities and the faculty & cadets at the Academy provides opportunities that would otherwise not be available to an undergraduate institution.

In addition to linking research between USMA and ARL, the ARL Liaison directs the ARL visiting scientist and Davies post-doctoral fellowship programs. Davies Fellows are post-doctoral fellows assigned for three years to West Point and sponsored by ARL. Davies Fellows conduct research with a scientist at ARL (coordinated and approved through the National Research Council) for half of their assignment at USMA and teach cadets for the remainder. Davies Fellows have the opportunity to sample the teaching and research aspects of a university while enjoying the benefits of integrating fully into the West Point experience.

Each year ARL sponsors a visiting scientist at USMA. The visiting scientist is on loan from ARL to interact with cadets in the classroom as well as conduct research in his/her particular specialty. The visiting scientist is treated as a visiting professor at the Academy and integrates into the senior faculty. The ARL visiting scientist program is yet another REACHING opportunity available in the Math department at USMA.

The Way Ahead

The MSC is committed to enhancing the tradition of excellence that the Department's faculty brings to the classroom and to individual cadets through various teaching and outreach programs. The Department provides a solid intellectual foundation in problem solving to every cadet through the core mathematics experience. The MSC and the Department's outreach program complement this effort by bringing current and relevant experiences into the classroom.

Future directions for the MSC include longer-term research projects outside the Academy as well as increasing the interdisciplinary approach to research by working with other departments. The MSC is also attempting to broaden research horizons by placing an increased emphasis on integrating more civilian faculty into REACHING opportunities. This effort will pay huge dividends in the classroom by demonstrating to cadets that our faculty is current and engaged in timely topics, thereby adding relevance to their educational experience.

These outreach initiatives are extremely important, but the Department's primary task is to educate cadets and prepare them for the rigors of their chosen major and the Army. Excessive outside activities can potentially detract from this success, so the Department leadership is careful to evaluate the initiatives we pursue, ensuring they are in line with the Department's mission. Ideally, this approach fosters an atmosphere that creates confident, competent problem solvers who are prepared to excel in their academic major and in the Army. ■

"Research vs. Teaching" or "Teaching and Research" ?

Dr. Heather Dye, USMA, Department of Mathematical Sciences

Teaching and research bring to mind two diametrically opposed personality types. On the one hand, we envision researchers to be isolated individuals in white lab coats surrounded by costly technical equipment who enjoy being alone and thinking about strange, remote concepts. On the other hand, we conceive of teachers as being surrounded by chalkboards and large groups of noisy children. Although these images are sometimes valid, they are stereotypes that have an unfortunate effect in mathematics. The message created by these images is that research and teaching have little in common. These stereotypes reinforce the notion that "research (especially in mathematics) is for the few while teaching and the classroom are for everyone else."

The reality is that for a mathematics professor, both teaching and research are important scholarly activities and they benefit both the student and professor. A math professor must ask two questions to realize this benefit. *How do my students benefit from my research? How does my research benefit from teaching students?* To answer these questions, we must consider, fundamentally, what it means to do research. We must also consider what knowledge, skills, and attitudes we hope to impress upon our students.

When I do research, I am a problem solver. I attempt to answer questions and fit my research results into the framework of existing mathematics. Answers evolve slowly, beginning typically as a large, amorphous idea: "I think this is what is happening." Then I refine this vague statement by narrowing the scope of my original questions. Such smaller questions often contain restrictions that make them easier to answer. The purpose of these questions is to provide evidence for or against my idea in a structured manner. One of my favorite strategies is to try to find examples or counterexamples.

This process is the same one I used to study for my classes as an undergraduate student. I did not ask the questions as an undergraduate student; initially, the questions were both given in their original form and refined into smaller questions for me. For example, the question "How do we solve differential equations?" was simplified by restricting the type of differential equation to first order, linear, homogeneous. I would work several examples and determine the mathematical steps involved in solving the equation.

I want my students to have these skills. I want my students to be able to ask a question and incorporate additional restrictions (or simplifying assumptions) into the question. I would like my students to have the ability to identify key examples and determine the necessary steps to solve these example problems. Rote pedagogical methods will not give this ability to my students. By using guided research exercises in class and as homework, I encourage my students to follow this process. I emphasize the importance of examples, guided experiments, and asking questions.

Most students understand the importance of examples, but they have difficulty understanding how to use them. They often try to fit problems into an inappropriate or unhelpful context. Programs such as *Mathematica* and Maple can quickly compute solutions and display a wide variety of graphics. They allow us to ask for suggestions from our students – and to try out the mathematical suggestions without stumbling through calculations. For example, when students suggest using a 2^{nd} order differential equation to model mixing, you can rapidly try their suggestions in *Mathematica* and demonstrate that a 2^{nd} order equation is not a good choice.

With a computer, I can present graphs, direction fields, and other examples without the time-consuming process of drawing a reasonably accurate picture on the chalkboard. These examples assist the students in building a mathematical awareness. For example, to determine whether a particular graph is a reasonable rendering of $f(x)=x^6$, we rely on our knowledge of the appearance of the graph of $f(x)=x^{2n}$. I know that the graph of x raised to an even power passes through the origin, is symmetric, and U-shaped. By showing my students a series of graphs, I can guide them to this realization. This process is more effective than simply stating that these graphs have characteristics 1, 2, and 3 without showing any supporting evidence. The statement of these characteristics presupposes that the students already know what "symmetric" and "U-shaped" mean in the context of graphs. Of course, if I need to tell the students about these characteristics, they probably do not understand these words in this context.

Textbooks often present formulas as the 'solution' but do not motivate the formulas or provide any developmental background. Such background may be incomprehensible to the student, and/or the supporting computations may be difficult to carry out. A series of carefully chosen examples can illustrate the use of a theorem or allow the students to restate a theorem in their own words. This process does not constitute a mathematical proof, but is certainly more believable to the students than simply saying "use this formula."

I would like my students to develop the ability to ask clearly articulated questions based on the lesson material. This is part of my research process, and this is the ability that will allow my students to become astute problem solvers. *This is also the most difficult task for students – to be able to ask the right question.* The right question will point out some aspect of the problem that will illuminate potential answers to the question. This process takes practice, and students need assistance in moving from a general question to a more specific question. When planning my lessons, I try to find the questions I would ask while researching. Which example best illustrates this concept? What aspect or process allows me to understand and interpret my mathematical result? Is there an example of this mathematical model in my everyday life?

I struggle with my research in much the same way that my students struggle with their lessons. This struggle constantly reminds me of my own experiences as a student. Mathematics professors teach mathematics: teaching students to read actively, to think in a logical and goal-oriented manner, and to present technical information. This reminder alters my viewpoint; it makes me stop and consider the mathematics from the student viewpoint.

Teaching provides the researcher with some important benefits. Without the greater mathematics community, research would have no applications and no framework. Students are part of that mathematics community, and they are the potential mathematicians, scientists, and engineers of tomorrow. Even outside the sciences and engineering, today it's difficult to find a career that doesn't require some level of mathematical sophistication.

Teaching provides a valuable opportunity to interact with the mathematics community. Teachers present mathematical material that is familiar to them, but it is up to the individual to present the material with a sense of relevance and interest. I often find details and other aspects that clarify the lesson topic while preparing for class. This new understanding connects with and clarifies other elements of my mathematical knowledge. Obtaining this understanding is fun and could possibly benefit my research interests. When I teach, I am practicing all the skills needed to do research. A teacher must communicate mathematics in both oral and written form, attempt to draw attention to the key ideas and concepts, and design examples and counterexamples.

It is important to remember that mathematicians don't spring from the sea foam. One of your students could be talking to you about research in a few short years. Without new mathematicians, there would soon be no reason to do any math. Teaching reminds me of my original interest in mathematics. Research keeps me involved in exciting new developments in mathematics. Teaching and research are not in competition but they are complementary aspects of a good math professor's work. ■

Recreational Research?

Dr. Michelle Ghrist, USAFA,
Department of Mathematical Sciences

"Work consists of whatever a body is obliged to do, and play consists of whatever a body is not obliged to do."

Mark Twain,
The Adventures of Tom Sawyer

I am obliged to conduct research; I must admit that it is work to me. I am also obliged to teach and interact with students; however, I find that this part of my job is more like play than work.

What is behind this dichotomy? Did it start back in my graduate school days when upon seeing entire rooms filled with journals, I would ask myself, "How can I ever contribute something new? And even if I thought that I did, how will I ever know that someone hasn't previously done the exact same thing, and it's sitting in a journal somewhere?" Do I favor teaching over research because I enjoy tasks that have well-defined beginnings and ends? Is it due to my being more of a work-out-the-details person rather than one to come up with great innovative ideas?

Whatever the reason for my early aversion to research, it played a great role when I chose my first faculty position at a small private liberal arts university which had no research or publication requirements for the faculty. At first, I felt great relief. However, after about a year or so without doing research, I was very surprised to find myself actually missing it! While I loved crafting a well-thought-out learning experience for my students in the classroom, something in me missed the challenge and excitement of forging new ground, of trying to put together the puzzle pieces to do something which had never been done before.

Thus, I began the job search process yet again and eagerly accepted my current position two years ago – USAFA seemed to offer the perfect balance of teaching and research for me. However, I am such a conscientious teacher that I found myself always putting my students first. When I finally was able to devote time to research, I spent weeks pursuing one great idea after another, and they all seemed to lead nowhere! I was beginning to feel like a failure again – what had made me think I could contribute to the vast array of original research?

Finally, over Spring Break this year, I was able to prove a conjecture that I had formulated a few years back. Success! When classes resumed the following week, I excitedly shared my result with my Numerical Analysis classes; we just happened to be studying some similar ideas. Over the next few weeks, I found myself trying to vocalize to my students some of my experiences regarding research. After some reflection, I believe that these are good lessons to show to **all** of our students regarding the relationship between teaching and research.

Research usually does not provide instant gratification.

I made my conjecture in 1998; I didn't have time to seriously explore it until 2003, and I finally proved it in 2004. Some conjectures have waited hundreds of years to be proved. (As a side note, I'm still searching for the elusive "obvious" proof of Fermat's Last Theorem, which Andrew Wiles proved in 1995, over 300 years after it was conjectured by Fermat.) There are still many conjectures outstanding.

It is rare that someone begins to explore a research idea and doesn't run into (seemingly) dead ends. While these setbacks are frustrating, they are often necessary for one's subconscious to work. In addition, the excitement one feels when success finally comes is so much more fulfilling because of the delays. It is in these times that research begins to be more like play than work.

Students get used to the pre-processed problems that textbooks and instructors assign to them. What do students take away from these problems? Often the message is "get the right answer" rather than understanding the process or the underlying fundamental ideas. This is not how research (or life) is. By providing students with more open-ended questions to explore, they start to see a glimpse of what mathematics truly is.

Detours are inevitable, necessary, and sometimes beneficial …

Many times, when we are heading down what seems to be a dead end, our paths often invite us to open up more doors than we would ever imagine. My many experiences (also known as detours) in life have led me to have a tremendously broad array of mathematical interests, including numerical methods, mathematical modeling (including biomathematics, geomathematics, and computational physiology), mathematics education (including curriculum and placement issues), mathematical poetry, mathematical philosophy, and humanistic mathematics (connections between mathematics and the humanities). There are some negative aspects to this, such as having trouble focusing on which areas to do research and my husband complaining about just how many books I have! But, there are some wonderful aspects also, such as being able to incorporate these broad interests into the classroom, and never being bored regarding research – like a magician, I always have one more trick up my sleeve!

Mathematics research often jumps across many boundaries that we artificially place on the field.

For the proof of my theorem that I did over Spring Break, I began with a question regarding the stability of numerical methods for solving initial value problems. By the time my proof was done, I had used the following mathematical tools: complex roots of nasty polynomials, continued fractions, Descartes' Rule of Sign Changes, the gamma, digamma, and polygamma functions, Euler's constant, double factorials, the binomial series, sequences (including L'Hopital's Rule), mathematical induction, a neat bounding argument (thanks to my husband), and *Mathematica.* I am fortunate that my mathematical background is quite broad (in both applied and theoretical mathematics); this allows me to come at problems from many different angles.

Students need to realize that real-life problems aren't confined to one area of mathematics. So often, they view mathematics as compartmentalized; we all have heard comments like, "I am done with algebra; why should I have to remember it to do Calculus?" Because research (and life) is not limited by artificial constraints, we need to (try to) teach students many different mathematical tools and approaches. We should also try to get them to expand their minds and try to combine these tools in innovative ways.

Research (like mathematics) is often like a puzzle.

You just need to figure out how to put all of the pieces together in the right order. In classes, we often provide students with most of the pieces (with many of them already pre-connected). What makes research so much harder is finding the right pieces. They may be hidden in journals, textbooks, or even in some of our chicken scratches. I even found one of the pieces I needed for my proof in an old high school mathematics book I had in my office! Sometimes others can help point us in the right direction in our treasure hunt to discover the right pieces. This is why collaborating with others can be so beneficial.

To do effective research, you need to know your strengths and weaknesses.

My mind usually does not make broad sweeps; it likes playing connect-the-dots. Thus, I am not very good at strategy games like chess, where one needs to be able to think five or so steps into the future to see how things will turn out. I am very good at extending older results to new ideas. By knowing my strengths and weaknesses as a researcher, I am able to collaborate with others whose strengths complement my weaknesses (and vice-versa), thus making our results that much better. Through the years, I have also learned from my collaborators how to overcome my weaknesses.

"If I have seen further, it is by standing on the shoulders of giants."

Isaac Newton, 1676 letter to Robert Hooke

Regardless of how we individually feel about research, the more relevant issue at hand is how do we represent research to our students? Do we provide students with authentic learning experiences that allow them to get a glimpse of what mathematics truly is? Do we allow them to experience the joy of discovery inside and outside the classroom? Do we set aside time for them to see the human side of mathematics by teaching them about the real people behind the results that they learn? Do we attempt to integrate what they are learning with the other "compartmentalized" areas of mathematics that they already know or will learn about in the future? More importantly, do we convey a love of mathematics and a love of learning in the classroom? I maintain that it is our obligation as teachers, as researchers, and as human beings to present the humanity of mathematics to our students. I believe that integrating teaching and research in the classroom is an ideal way to do this. We need to realize, and convey to our students, that none of us needs to be a giant in and of ourselves to enjoy the wondrous view from the shoulders of giants.

"A dwarf on a giant's shoulders sees farther of the two."

George Herbert, Jacula Prudentum, 1651 ■

Personal Research Enhances One's Own Mental Condition and Relationships with Students

Prof. Brian Winkel, USMA, Department of Mathematical Sciences

I have one research topic I pick up every once in a while that accomplishes several goals in my personal and professional life: it engages my thinking and it interests my students. I have used this topic some five or six times over a 20-year period to get students to discover and see how to do research for themselves. I have not yet published any results in the area. This is an easy topic to get into and to explore and I have no trouble "snaring" students into considering it, nor do they have trouble seeing patterns, asking questions, or getting results.

I call this *personal research* because the issues are on the order of "What is going on here?" "What are these things like?" "What might we do with them?" "How pretty are they?" I know of no applications; I have only seen one article [2] on this topic and even that author knows little about its history or applications.

I have found that students are immediately captivated by the topic and I have had great success in engaging their minds to discover patterns and principles in many individual directed (or non-directed!) studies over the years. I once had a team of three students buzzing over the material and just recently I had a bright student examining possibilities [1]. In this most recent round I discovered things I did not know about the phenomena.

So what is this stuff and what is so engaging about it? Well, it is geometric so it permits pictures. That helps! It is about familiar functions. That helps! It has a tactile sense. That helps! It is easy to explain and one can quickly get feedback and graphical outputs for stimulation. That really helps!

THE STUFF: BIPOLAR FUNCTION PLOTS

The phrase "bipolar function plots" names a methodology for producing plots. See Figure 1 (from [1]).

1. Set two poles, $P_1 = (0, 0)$ and $P_2 = (1, 0)$.

2. Consider a function $s = f(t)$.

3. At pole P_1 construct a ray at the angle of t radians measured counterclockwise from the x-axis and at pole P_2 construct a ray at the angle of $s = f(t)$ radians measured clockwise from the x-axis.

4. Now find the point of intersection $P(t) = (x(t), y(t))$ of these two rays. This point is the parametric value of the bipolar function and the collection of such points is the bipolar function plot for the function $s = f(t)$.

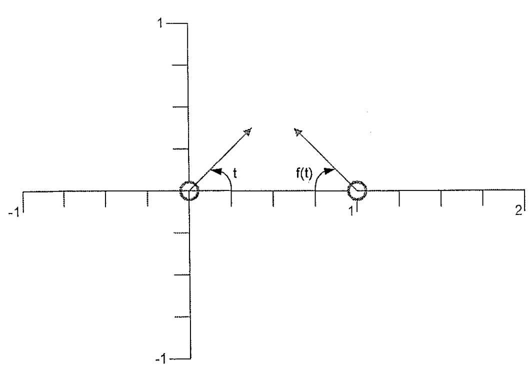

Figure 1. Generating Bipolar Function $s = f(t)$.

Immediately students wonder what obvious functions like $f(t) = t$ or $f(t) = mt + b$ or $f(t) = \sin(3t)$ will produce as a bipolar function plots. Indeed, $f(t) = t$ is obvious – see Figure 2.

Over the years I let the students discover appropriate ways to represent this process and they come to the conclusion that the point $P(t) = (x(t), y(t))$ has a nice parametric equation representation. They do the geometry, they build the equations. I give them nothing. Here is what they obtain:

$$x(t) = \frac{\sin(f(t)\cos(t))}{\sin(t + f(t))} \quad y(t) = \frac{\sin(f(t)\sin(t))}{\sin(t + f(t))} \quad (1)$$

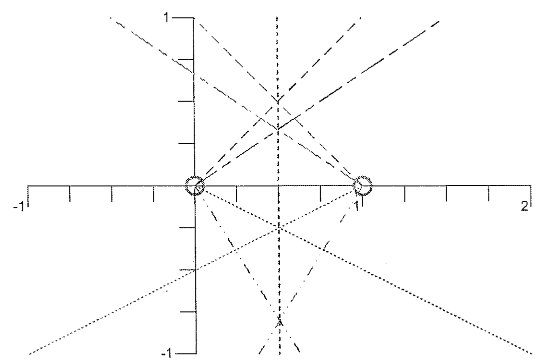

Figure 2. Bipolar function plot for $f(t) = t$. From [1].

Once they have the representation of the bipolar function as a set of parametric equations offered in Equations (1) then it is easy to use technology to render plots and begin experimenting. I have used both *MathCad* and *Mathematica* and the animation features in both to explore families of functions, e.g., $f(t) = at$ to study the effects of changing parameter a on the bipolar plots.

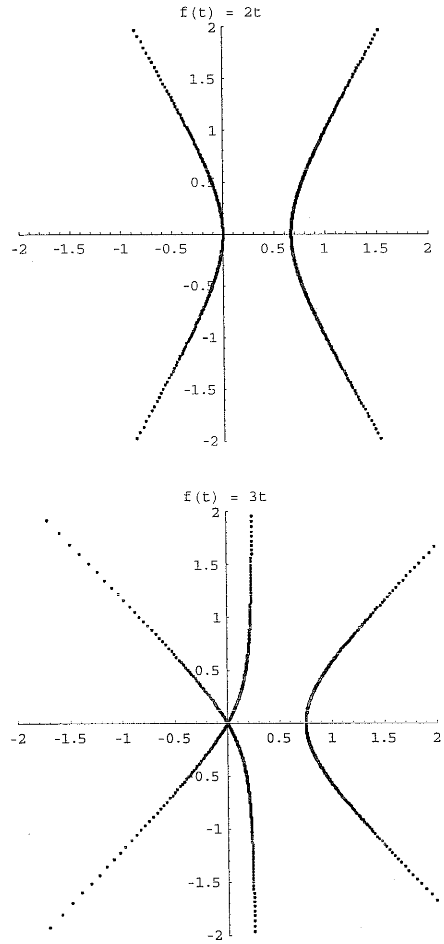

Figure 3. Bipolar plots (upper) $f(t) = 2t$ and (lower) $f(t) = 3t$.

Figure 3 shows the plots using *Mathematica*'s ListPlot command in which "connect-a-dot" action is not in force, while Figure 4 shows *Mathematica*'s ParametricPlot routine in which the connect-a-dot feature indicates the asymptotes. You should be able to see some pattern emerging in just these three plots. Students do! Issues emerge right away in just the study of the family $f(t) = nt$ for $n = 1, 2, 3, \ldots$ Where does this asymptotic behavior come from? Can we predict the number of asymptotes and where they will occur?

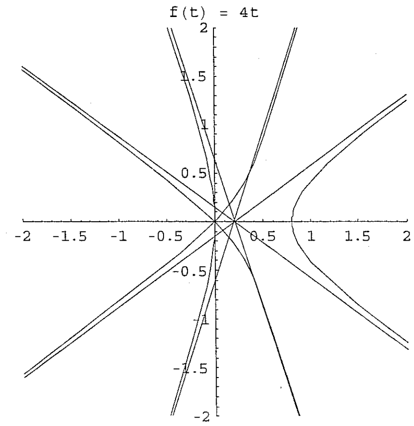

Figure 4. Bipolar plot $f(t) = 4t$.

Do the asymptotes intersect – they appear to do so in Figure 4 – and if so where? Students often will move slowly but an occasional student will want to move to $f(t) = mt + b$ right away. See Figure 5.

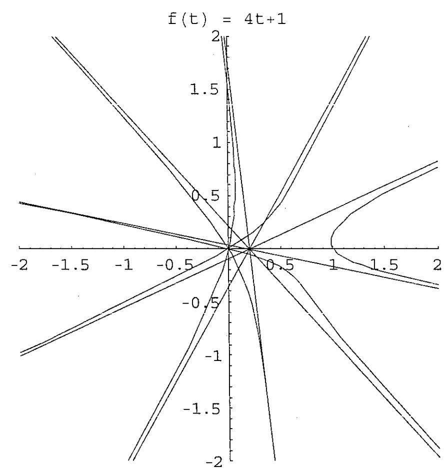

Figure 5. Bipolar plot $f(t) = 4t + 1$.

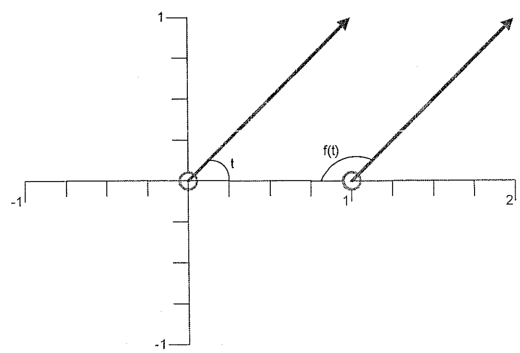

Figure 6. Diagram showing when asymptotes will occur; i.e., when angle t and angle $f(t)$ are supplementary.

This research is good practice in visualization and geometry for noting when asymptotes will occur. From Figure 6 we see that asymptotes will occur when we have $f(t)+t=\pi$, actually $f(t)+t=n\pi$, where $n=1,2,3,\ldots$

One can study entire families of functions, e.g., $f(t)=\cos(nt)$; $f(t)=t^n$, as well as parametric families $F(s,t)=s^2+t^2=25$.

INVERSE PROBLEMS

Another rich area of exploration is to "predict" what relationship is needed between the bipolar angles t and $s=f(t)$ – more generally, suitably specifying $F(s,t)=0$ – to produce a given shape when used in the bipolar sense. I had never thought of this until when, one year, a student simply asked, "I wonder what function would produce a circle as a bipolar function plot?" One can use solvers and implicit plot routines in *Mathematica* to produce the circle of radius 0.1 with center (0.5, 1) as a bipolar function plot (lower portion of Figure 7) when the relationship between bipolar angles t and s is given by the implicit function seen in the upper portion of Figure 7.

In Figure 7, the point (1.18, 1.09) as a parameter (t, s) will produce the point on the upper left portion of the bipolar function plot image circle while the point (1.05 , 1.20) as a parameter (t, s) will produce the point on the upper center portion of the image circle.

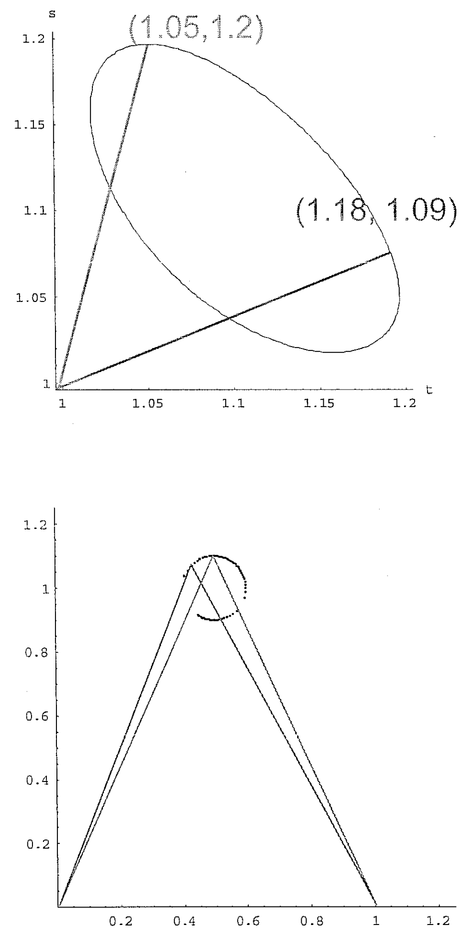

Figure 7. Angles (t, s) in the relation $F(t, s)=0$ (upper) will produce the circle of radius 0.1 centered at (0.5, 1.0) as a bipolar plot (lower).

RESEARCH VALUE

Students discover all these things and many more. They conjecture, they make lists of observations, they see the difference between (for n small or large) observations and a proof for all values of n, and they ask questions. For example, "What does the bipolar derivative, $f'(t)$, mean?" "How many asymptotes, petals, relative max/mins, etc. are there for a given function?"

I am compiling a catalog of events, of results, of observations proved, of curves, and of questions and unsolved issues. Each time I introduce students to the subject I let them discover the elementary principles for themselves and in every instance they go in their own direction discovering new things. I produce tools to enable student discovery, some useful and some not so useful, e.g., projecting plots from the plane onto the sphere so that asymptotes will map to finite curves through the north pole of the sphere. See Figure 8.

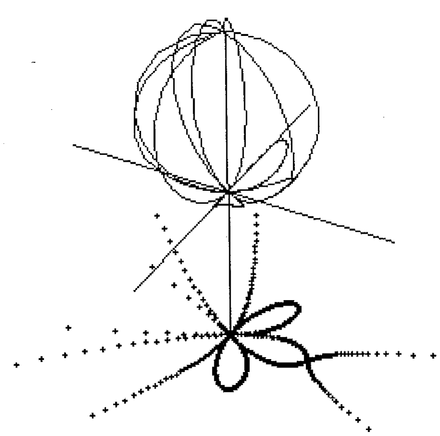

Figure 8. Depiction of the bipolar function $f(t) = \cos(3t)$ in the xy- plane and its projection onto a sphere raised several units above the plane for clarity.

This area of investigation, bipolar functions, gives students a taste of research as they wander the geometric, algebraic, and calculus landscape in search of interesting things to study, of principles and general statements to extract, and of families of functions to organize and relate.

If you have something you have always wondered about then share it with your students as a joint research effort. Both you and they will grow because of the sharing.

ACKNOWLEGEMENT

Figures 1 - 7 used in this article come from the final presentation of Ann Millen, a senior cadet at West Point, in her Senior Seminar MA491 directed effort which I supervised. ■

REFERENCES

[1] Millen, Ann. 2004. Patterns in Bipolar Parametric Plots -- Projects Day Presentation, 5 May 2004. West Point NY: US Military Academy.

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Research and Intuition

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The following is a well-known formula, but the quote, as cited in [2], is perhaps not so well-known.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad (1)$$

> *A mathematician is one to whom that (1) is as obvious as that twice two makes four is to you. Liouville was a mathematician.*
>
> -- Sir William Thomson, First Baron Kelvin of Largs.

One cannot help but feel good for being a mathematician after reading the above quote. After all, if a man such as Lord Kelvin, a well-known Englishman who was primarily a physicist, spoke well of mathematicians and more in particular a French mathematician, then we indeed all have great cause to rejoice at our own profession. Deriving (1) takes nothing more than a little knowledge in multi-variable calculus; some *ad hoc* proofs require even less. Although a truly rigorous proof needs some convergence arguments in the execution, the ideas behind (1) are not deep.

When I was in graduate school, I was always amazed and impressed with the vast amount of mathematical knowledge my thesis advisor, Professor Henryk Iwaniec, had and how he could put complicated mathematical theorems into simple words. In brief, I enjoyed his lectures because he revealed not only mathematical theorems, but also mathematical intuitions behind the proofs. He made mathematical ideas stand out like "a simple and clear-cut constellation," not some star hidden in "a scattered cluster in the Milky Way" veiled behind the enunciation of the statement and proof of the theorems ([1]). Indeed, mathematical ideas have less to do with the formality of the proofs; rather, the basis of intuition upon which the proofs are executed is the key for making mathematical discoveries. Having such intuitions is generally the mark of a good mathematician, and was perhaps partially the reason Lord Kelvin made the above remark. For a mathematician like Joseph Liouville, it was the mathematical intuitions he had that made (1) as simple and clear as $2 \times 2 = 4$.

It is generally believed that intuition and rigor are two integral parts of mathematics. Intuition tells us where to go and rigor gets us there. Hence, it is in this sense that mathematical intuition serves as a guiding light in problem solving and having it or not is, for the most part, the touchstone for mathematical sophistication. We, as mathematical educators, must be judged by a higher set of standards than our students and we would find ourselves severely handicapped if our own levels of mathematical sophistication did not exceed the bare minimum of what is needed to carry out our teaching duties. We all had the experience of looking back at the problems in elementary school while in high school and realizing that the arithmetic problems that troubled us before had become matters of triviality. It was not because we had been constantly practicing these arithmetic problems throughout our four years of high school that enabled us to do them with ease, but rather because we were working at higher and higher levels of mathematical stratum. Indeed, we had grown over those years and gained much mathematical sophistication so that, in high school, the problems from fourth grade weren't so hard anymore because we knew exactly what to do when we saw them. I claim that to be able to accomplish the similar feat at a university level and solve problems and explain concepts well, we must raise the bar of mathematical sophistication even higher and set it at the level of original mathematical research. It is indeed an oxymoron, but it takes experience to do things *a priori*, *id est* doing things with intuition. Mathematical intuition, the best indicator of mathematical sophistication, takes practice to attain. Such practice, for us, is mathematical research.

It is through this that I believe mathematical research reinforces teaching, but not in a direct way. I have yet to find any direct significant relevance that my research in the theory of exponential and character sums would have in any course I have taught thus far. But research does help me gain mathematical sophistication, and that gain facilitates deeper understanding of the material I must teach. One of the luxuries that a mathematician has in his work is that he never has to verify his motivations but only his results. It is this luxury that sometimes does a disservice to mathematics and makes much of the best of mathematics appear *ad hoc* and almost magical to novices such as our students. This phenomenon is present in both university mathematics and higher mathematics. I have been asked by a curious cadet what the motivations of the definition of the determinant of a matrix were. That definition is truly quite *ad hoc* upon first glace and the motivations are quite well-concealed. Indeed, definitions are mere assignments of names and we can call whatever thing whatever name we want. But if such assignments are to be meaningful at all, we must be somewhat motivated to make the assignments. To fully explain the motivation of the definition of the determinant of a matrix, one is unlikely to do a satisfactory job without some deeper understanding and intuition in the theories of matrices and linear operators. That kind of understanding and intuition is generally attained through mathematical research.

Often I hear my cadets asking questions such as "Is that the final answer?" or "What does that (the solution to a mathematical model) tell us?" We certainly should not expect our cadets to have the kind of mathematical sophistication found in research mathematicians. Indeed, some computations tend to be so lengthy that it is easy to lose one's direction along the way. For these reasons, the two questions above are perhaps somewhat excusable. But I believe that it is precisely the lack of mathematical intuition that prompts the two questions from cadets. My response to the first question has often been "What was the question we started with? Have we addressed it yet? If so, then we have the final answer; otherwise, no." In the similar manner, my response to the second question has been "What does our mathematical model represent? What do you think the solution to that model should represent?" It is certainly not the most courteous thing to do to answer questions with questions. But at least it is hoped that they will put some thought into the problem solving process.

I feel that helping my students gain mathematical sophistication or intuition is among the most important tasks I have as a mathematical educator. As noted in the last paragraph, such intuition is very much needed by many of my students. They seem to be lost in the middle of problem-solving and, as mentioned before, mathematical intuition serves the very purpose of telling one where to go. Consequently, to impart the kind of intuition that is so needed by my students, it is imperative that I must have a higher level of mathematical sophistication myself. As I have pointed out earlier, I believe that such sophistication is attained through mathematical research.

However, I believe that the reinforcement in the other direction is not as strong. At least, I have not yet seen in any significant way how my teaching has helped me in my research. I feel that it is simply because the material taught in early undergraduate mathematical education (all the courses I have taught are freshmen or sophomore courses) hardly rises up to scholarship standards that are commonly recognized in mathematical research. The closest example of my teaching serving to reinforce my research is the technique for converting a repeating decimal into a fraction of integers. This arose while teaching a pre-calculus class. I later realized that such techniques were precisely what were needed to re-prove a lemma of Gauss. But of course, that was certainly NOT original research and neither was it a deep lemma by today's standards. It is certainly conceivable that teaching can reinforce research, but my experiences have been that the reinforcement goes essentially only one way. ■

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[2] M. Spivak, Calculus on Manifolds, Addison-Wesley, New York, 1965.