PRODUCT-FREE SETS IN THE CARD GAME SET

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ABSTRACT: The card game SET has attracted the attention of math and game enthusiasts alike. In this article, I present a first semester Abstract Algebra project that guides the students through an algebraic formulation of the game. There are many interesting mathematical questions that one can ask about the game, and I illustrate how the project can be used to get students working on such questions. In this way, the project can also serve as an opportunity for undergraduate research.

KEYWORDS: Abstract algebra, product-free sets, binary operation, undergraduate research

INTRODUCTION

The card game SET is known to many as an addictive pastime requiring speed in pattern-recognition. The game was invented in 1974 by population geneticist Marsha Jean Falco, who used symbols on cards to represent genetic data of German Shepards in an attempt to better understand epilepsy in the animals. Falco's search for genetic patterns ultimately led to an award-winning game. In the early 90's the game received honors from MENSA (one of the top five games, 1991), *Omni* Magazine (one of the best games of the year, 1991), and *Games Magazine* (one of the top 100 games for 1992, 1993, 1994, and 1995). Since that time, SET has continued to raise interest, and most recently, this interest has come from within the mathematics community. In the summer of 2003 Ivars Peterson wrote an article [4] for the *MAA Online* summarizing some of the mathematics involved in the game. He was inspired by a nice article that appeared in the *Mathematical Intelligencer* [1].

As a long-time SET enthusiast myself, I too became interested in the mathematical structure of the game, and in the Spring of 2003 (before the two articles cited above actually appeared) I designed a project for my Abstract Algebra class that guided the students through an algebraic formulation of the game. My hope was that the project would spark the students' interest in Abstract Algebra while solidifying their understanding of some of the basic concepts I had already covered in the course: binary operations, direct products, permutations, and others. I also assigned an optional "Challenge Problem" (which I myself had not solved) with the additional thought that the project might provide an opportunity for undergraduate research.

As I discovered, many of the students were already quite familiar with SET, and this familiarity helped to ignite their interest in the project. However, there were several students who had never played the game, so I introduced the project by explaining the basic rules and object of the game. For the reader unfamiliar with SET, a description is given below.

The Object of the Game

The object of the game is to identify a "Set" among an array of 12 cards laid on the table. Each card has four characteristics: symbol, color, number, and filling, and each of these four characteristics can take on one of three possible values:

1. **Symbol**: Each card contains either ovals, squiggles, or diamonds.

2. **Color**: The color of the symbols are either red, green, or purple.

3. **Number**: There are either 1,2, or 3 symbols on each card.

4. **Filling**: The symbols on each card are either filled in (solid), unfilled, or striped.

A collection of three cards is said to be a "Set" if, for each of the four characteristics, all three cards either share the characteristic or each is different. For example, the three cards shown in Figure 1 below form a Set because the symbols, number, and shading are all different on each of the cards, and all three cards share the same red color (although the black and white image does not allow the reader to see this.)

On the other hand, the three cards shown in Figure 2 do not form a Set because only two of the three cards share the same (striped) filling.

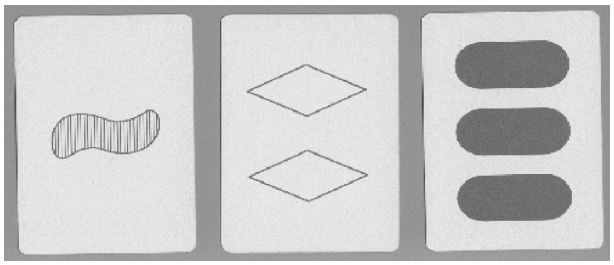

Figure 1. Assuming that the symbols on all three cards are red, the cards above form a "Set". All of the characteristics except color differ between the cards.

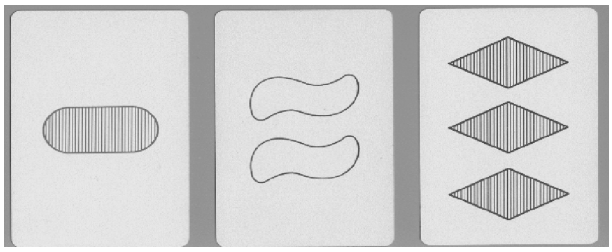

Figure 2. Three cards that fail to form a "Set" because only two of the three cards share the same striped filling.

THE MATHEMATICAL FRAMEWORK

The Labeling Scheme. Since there are 4 characteristics, each of which has 3 possible values, we can associate each card in the SET deck with an element in the set $D = \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. We simply assign a 1, 2, or 0 with each of the three possible values for a given characteristic. There are certainly many different ways to do this, so we will pick the one described in the table below and stick with it.

Symbol	Label
oval	1
squiggle	2
diamond	0

Color	Label
red	1
green	2
purple	0

No.	Label
one	1
two	2
three	0

Filling	Label
solid	1
unfilled	2
striped	0

Finally, we will agree that the four summands of $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$ will serve as a labeling system for the four characteristics: symbol, color, number, and filling *in that order*. So the first summand describes the characteristic "symbol", meaning that a card of squiggles will be identified with a 4-tuple having a 2 in its first entry. (See Figure 3 below for examples of the labeling scheme.) Note that every possible combination of the four

characteristics does, in fact, appear in the SET deck. Hence, there are a total of 81 cards in the deck – exactly the same number as the order of $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$.

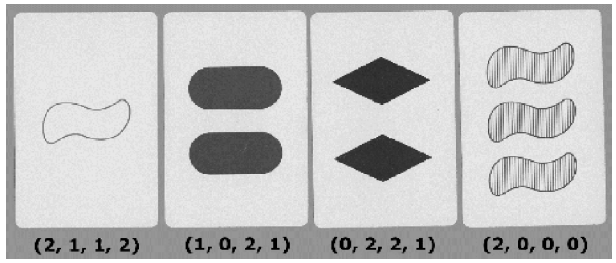

Figure 3. Four cards and their corresponding elements in $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. The first card is red, the second and fourth purple, and the third card is green.

The Multiplication on D

Next we will define an operation on $D = \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. The operation will be chosen to reflect how one obtains a "Set" from two given cards. If $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are two elements of D, then define xy as follows:

$$xy = (2(x_1 + y_1) \bmod 3,\ 2(x_2 + y_2) \bmod 3,\ 2(x_3 + y_3) \bmod 3,\ 2(x_4 + y_4) \bmod 3)$$

Note that $2^2 = 1 \bmod 3$, that is 2 is its own multiplicative inverse in Z_3, and so we could just as well define the product xy to be:

$$xy = \left(\frac{(x_1 + y_1)}{2} \bmod 3,\ \frac{(x_2 + y_2)}{2} \bmod 3,\ \frac{(x_3 + y_3)}{2} \bmod 3,\ \frac{(x_4 + y_4)}{2} \bmod 3 \right)$$

In this way, we can think of xy as the average of x and y in D.

THE PROJECT

The project in my abstract algebra class consisted of the following ten exercises. After a brief introduction to the game, I gave each of the thirteen students in my class a SET deck in the hope that this would excite them and encourage them to take a hands-on approach to problem-solving. I gave the students two weeks to complete the first nine exercises (while other daily assignments continued), and I offered extra credit on the tenth problem up until the last day of class. One student took me up on my offer!

Exercise 1. Suppose that $x = (1, 1, 1, 1)$, $y = (0, 0, 0, 0)$, $z = (1, 2, 2, 0)$, and $w = (2, 2, 1, 1)$. Compute each of the products xy, xz, xw, and zw

(symbolically), and then sketch in the appropriate symbols to see how this multiplication translates into cards. You will need green, red, and purple pens to complete this exercise! Describe what you discover.

Comments: As most students recognized right away, the product xy does indeed correspond to the card that completes the Set containing the two cards represented by x and y. To see that the operation is well-defined, simply note that $x_i y_i = \frac{(x_i+y_i)}{2} \bmod 3$ is a well-defined operation in $\mathbf{Z}_3$.

Exercise 2. Once you understand what it means to multiply two cards together, you might wonder if D is a group under this multiplication. Let's explore this issue now. Answer each of the following questions, providing proofs of each of your claims.

a) Is the multiplication on D associative?

b) Is the multiplication commutative?

c) Does D contain an identity element?

d) Do inverses exist in D?

Comments: Straightforward computations show that the operation is commutative but not associative, and that $xz = x$ holds true only when $z = x$. Hence D does not have an identity element, and it does not make sense to speak about inverses.

Exercise 3.

a) In light of what you discovered in Exercise 1, explain why you would expect the following two properties to hold true. (Think about what these properties say in terms of the cards.)

$$\textbf{Property 1} : x(xy) = y$$

$$\textbf{Property 2} : (xy)y = x$$

b) Now *prove* that these two properties do indeed hold true.

Comments: These properties reflect the fact that any two cards are in exactly one set. Since xy completes the Set containing x and y, Property 1 says that y must complete the Set containing x and xy, and Property 2 says that x must complete the Set containing xy and y. That is, $y = x(xy)$ and $x = (xy)y$.

Exercise 4. Does cancelation hold in D? That is, if $x, y, z \in D$ are such that $xy = xz$, then must $y = z$? Provide a proof to support your claim.

Comments: Approximately one third of my students recognized that cancelation follows from Exercise 3 after multiplying both sides of $xy = xz$ by x. The rest of the students used a "brute force" approach.

Exercise 5. Prove that if $x, y, z \in D$, then $(zx)(zy) = z(xy)$ (and similarly, $(xz)(yz) = (xy)z$.)

Comments: This turns out to be a somewhat tedious computation, but the result is useful and meaningful.

Exercise 6. A **D-set** is defined to be a subset $S \subseteq D$ of the form $S = \{x, y, xy\}$, where $x, y \in D$. Obviously, D-sets correspond to "Sets" in the card game. Prove that there are a total of 1080 possible D-sets in D (and hence there are a total of 1080 possible sets in the card game). How many D-sets can a given element $x \in S$ be a member of?

Comments: Most of my class had not taken a course in discrete mathematics and were not too comfortable with counting arguments. Although many of the students recognized that 1080 was a result of dividing $81 \cdot 80$ by 6, some needed additional prodding to understand how the 6 accounts for redundancies in the D-sets. This issue is relevant in the second part of the question too. While there are 80 possible cards y that can be paired with a given card x to determine a D-set $\{x, y, xy\}$, half of them will be redundant since x and xy determine the same set as x and y. Hence there are 40 different D-sets containing a given element x.

Exercise 7. Given $w \in D$, define the map $T_w : D \to D$ by $T_w(x) = wx$. Is T_w a permutation on D? Provide a proof to support your claim.

Comments: This result follows directly from Exercises 3 and 4.

Exercise 8. Suppose that $U \subseteq D$. Then we say that U is **product-free** if $xy \notin U$ whenever $x, y \in U$. Note that product-free subsets of D translate into collections of cards that fail to contain a "Set". Prove that if U is product-free, then xU is product-free too. (Recall that for any set $S \subseteq D$ and $x \in D$, xS is defined to be the set $xS = \{xs \mid s \in S\}$.)

Comments: Most students used a proof by contradiction here. Assuming that $\{xu_1, xu_2, xu_3\}$ is a D-set in xU, $(xu_1)(xu_2) = x(u_1u_2)$ by Exercise 5. Since there is exactly one D-set containing xu_1 and xu_2, $xu_3 = x(u_1u_2)$. Hence $u_3 = u_1u_2$ by Exercise 4, and therefore $\{u_1, u_2, u_3\}$ is a D-set in U. This, of course, contradicts the fact that U is product-free. Incidentally, I found this exercise (as well as the next one) to be particularly useful in getting students more comfortable with the notion of a coset. While the set xU is not actually a coset, it certainly is "reminiscent" of one.

Exercise 9. Prove that if $S \subseteq D$ is a product-free set and $x \in S$, then $S \cap xS = \{x\}$. Conclude that any product-free set can contain at most 41 elements. (This of course means that any collection of 42 cards must contain a "Set".)

Comments: This result was inspired by a similar result in [2, p. 901] about product-free subsets of groups.

Exercise 10 (An Optional Challenge). The largest known collection of cards containing no Set is given below. Its cardinality is 20 [5]. If we let $\alpha(D)$ denote the cardinality of the largest product-free subset of D, then this fact together with the result of the previous exercise yields $20 \leq \alpha(D) \leq 41$. Can you tighten the upper bound on $\alpha(D)$? That is, can you find an upper bound that is less than 41, or optimally, can you prove that $\alpha(D) = 20$?
A product-free set of 20 elements in D:

$$\begin{array}{lllllll}
(0,1,0,0) & (0,2,0,1) & (1,2,0,0) & (1,1,0,1) & (2,1,0,1) & (2,2,0,0) & (0,0,2,0) \\
(0,1,1,0) & (0,2,1,1) & (1,2,1,0) & (1,1,1,1) & (2,1,1,1) & (2,2,1,0) & (0,0,2,1) \\
(0,1,2,2) & (0,2,2,2) & (1,2,2,1) & (1,1,2,0) & (2,1,2,0) & (2,2,2,1) &
\end{array}$$

Remark: In [1], Benjamin Lent Davis and Diane Maclagan describe an equivalent formulation of the question posed above (as well as the parallel question for possible generalizations of SET). As they explain, the reformulation involves maximal caps in a four dimensional vector space over the finite field $\mathbf{Z}_3$. Interestingly enough, this equivalent question was answered by Giuseppe Pellegrino [3] in 1971 - before the SET game was even invented! Hence, $\alpha(D)$ is exactly equal to 20. Nonetheless, the mathematics involved in the proof is beyond the scope of a first-semester Abstract Algebra course. It would be nice to tighten up the upper bound on $\alpha(D)$ using the more elementary approach outlined in this project.

STUDENT RESULTS

Overall, the results were quite positive. I was pleased that students were able to work independently on these exercises, requiring little explanation from me beyond my initial 15-minute introduction. While I encouraged students to work together on the problems (and they did), I required that each student submit his or her own write-up. This allowed me to provide students with helpful feedback on their proof-writing skills. All of this work was done outside of class, and the result was a class average of 81.5% on this project.

Student response was favorable. When submitting their work, several students noted their appreciation of the project because it allowed them to gain a better working knowledge of so many abstract ideas. In the course evaluations at the end of the semester, many students cited the SET project as their favorite part of the course. They described the project as "fun" and "cool".

Perhaps most exciting, however, is the fact that one of my students took me up on my offer for Extra Credit (and of course, he was in no need of it). On the last day of class, junior math major Jun Ma submitted a nice result, lowering the upper bound for $\alpha(D)$ from 41 to 33. His work earned

him a Franklin Miller Award at Kenyon, an award honoring students for extraordinary work that goes beyond what is required in a course. I include Jun's proof below.

THEOREM. Let $\alpha(D)$ denote the cardinality of the largest product-free subset of D. Then $\alpha(D) \leq 33$.

PROOF BY JUN MA (KENYON COLLEGE, CLASS OF 2005): Let S be a product-free set in D, say $S = \{s_1, s_2, s_3, \ldots, s_n\}$, and consider the set $s_1S = \{s_1, s_1s_2, \ldots, s_1s_n\}$. By Exercise 9, $S \cap s_1S = \{s_1\}$, so certainly $S \cap (s_1S \setminus \{s_1\}) = \emptyset$. Now consider all D-sets constructed by picking two elements x and y in S. Since S is product-free, $xy \notin S$. Therefore, there are $n(n-1)/2$ sets that can be formed in this way.

Next consider all D-sets constructed by picking two elements $x, y \in s_1S \setminus \{s_1\}$. Clearly, $s_1S \setminus \{s_1\}$ is product-free by Exercise 8. Hence there are $(n-1)(n-2)/2$ possible sets containing x and y. Since $S \cap (s_1S \setminus \{s_1\}) = \emptyset$, the $n(n-1)/2$ sets formed by choosing two elements in S are different from the $(n-1)(n-2)/2$ sets formed by choosing two elements in $s_1S \setminus \{s_1\}$. And because there are 1080 possible sets in total (Exercise 6), we conclude that

$$\frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} \leq 1080$$

or equivalently, $n^2 - 2n + 1 \leq 1080$. Hence $n \leq 33.86$, and since n is an integer, we can conclude that $n \leq 33$. □

One might try to take Jun's argument a bit further, by also counting those D-sets constructed by picking two elements at random from the set $D \setminus (S \cup s_1S)$. Unfortunately, such an approach leads to an upper bound of 33.3 which does not improve the result.

PEDAGOGICAL BENEFITS

While I have already cited some of the pedagogical benefits of the SET project, there are others that deserve mentioning. Most importantly, I believe that this project is useful because it illustrates that Abstract Algebra is not just about clever tricks and abstract objects and properties that generalize the integers (as I fear many undergraduates believe.) Rather, Abstract Algebra is a rich set of tools that allow us to define useful frameworks on a wide range of situations, making computation and problem-solving possible - even easy. By defining such a framework on a game so familiar, I believe that the students see for themselves just how useful and concrete Abstract Algebra can be.

As a side benefit (and one that I had not anticipated,) I found that the project served to heighten interest in mathematics in students outside of

mathematics. Roommates and friends of students in my class were hearing all about SET and how mathematics could be employed to better understand the game. What a pleasant surprise it was for me to see this level of enthusiasm among the students!

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BIOGRAPHICAL SKETCH

Judy Holdener is an Associate Professor in the Mathematics Department at Kenyon College in Gambier, OH. She earned her BS at Kent State University and her MS and PhD at the University of Illinois in Urbana. She spent three years teaching at the United States Air Force Academy and is currently a visiting professor at the University of Colorado in Boulder, Colorado. Judy's research interests include algebra and number theory, and she enjoys getting undergraduates involved in mathematical research. Her leisure interests include gardening, hiking, and art, but she spends most of her time trying to keep up with her three-year-old son.