Harmonic Motion

Simple Harmonic Motion

Figure 1 shows a spring-mass system. In the unstretched position, the spring is suspended from a rigid body and no forces act on the spring. A mass *m* is attached and stretches the spring a distance *s*. The mass comes to an equilibrium position because the force of gravity acting on the mass, F_g, balances the restoring force of the spring, F_s. It is necessary to satisfy Newton's first law of motion to achieve equilibrium. The sum of the forces acting on a body at rest must equal zero.

$$\sum F = 0 \quad (1)$$

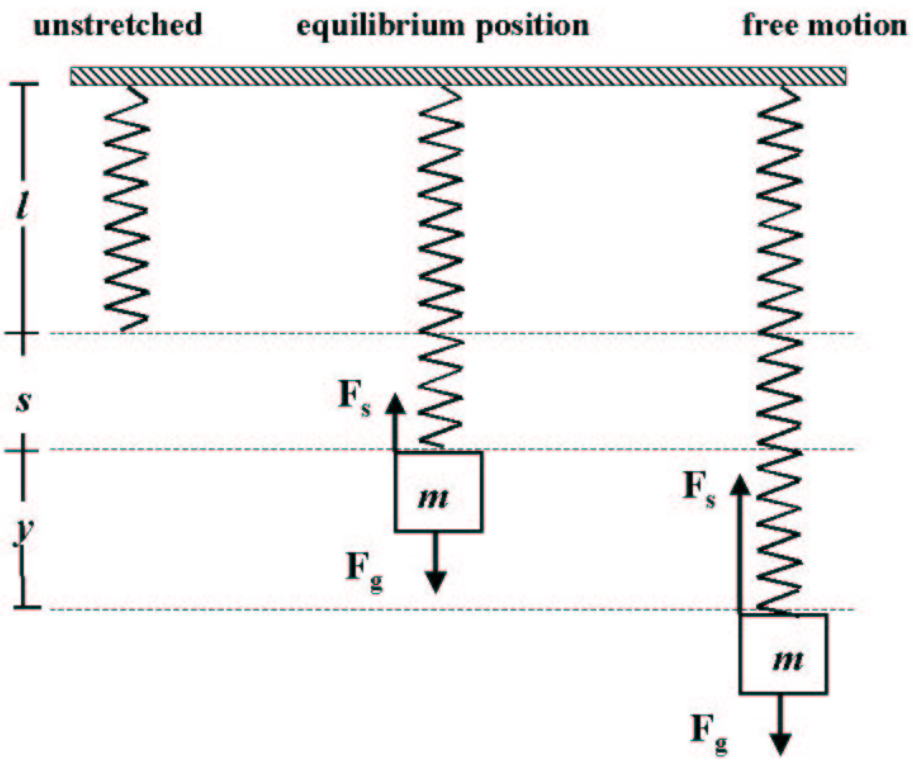

Figure 1

In the equilibrium position

$$F_g + F_s = 0 \quad (2)$$

where $F_g = mg$ and $F_s = -ks$ (Hooke's Law)

By substitution

$$mg - ks = 0$$

$$mg = ks \quad (3)$$

Equation 3 states that the force of gravity acting on the mass is equal to the restoring force of the spring.

If the mass is displaced a distance *y* from the equilibrium position and released, Newton's second law of motion gives the net force acting on the system, $F = ma$. F_s and F_g are not balanced so there is a net force, a resulting force that causes to the system to be in motion with a non-constant velocity. The dynamics of the system are governed by the equation of motion

$$\sum F = ma \quad (4)$$

The left side of equation 4 is the sum of the restoring force of the spring and the gravitational force acting on the mass.

$$\sum F = F_s + F_g \quad (5)$$

The acceleration, *a*, is the second derivative of the position function of the mass, $\frac{d^2y}{dt^2}$. Substituting into (5)

$$m\frac{d^2y}{dt^2} = F_s + F_g \quad (6)$$

Using Hooke's Law we can write a new expression for the restoring force of the spring, $F_s = -k(s+y)$. Assuming that there are no other external forces acting on the system,

$$m\frac{d^2y}{dt^2} = -k(s+y) + mg$$

$$m\frac{d^2y}{dt^2} = -ks - ky + mg$$

using equation 3 and simplifying we obtain

$$m\frac{d^2y}{dt^2} = -ky \quad (7)$$

$$m\frac{d^2y}{dt^2}+ky=0 \quad (8)$$

Equation 8 is a second-order differential equation for simple harmonic motion or free undamped motion.

Let $b = 0$ and explore simple harmonic motion using the applet.

Free Damped Motion

Consider a mass suspended in a viscous medium as shown in Figure 2. A damping force acts on the mass. Damping forces acting on a body are assumed to be linearly proportional to the instantaneous velocity, $b\frac{dy}{dt}$, where b is the damping coefficient. Modifying equation 7,

$$m\frac{d^2y}{dt^2}=-ky-b\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=0 \quad (9)$$

Equation 9 is a second-order differential equation for free, damped motion.

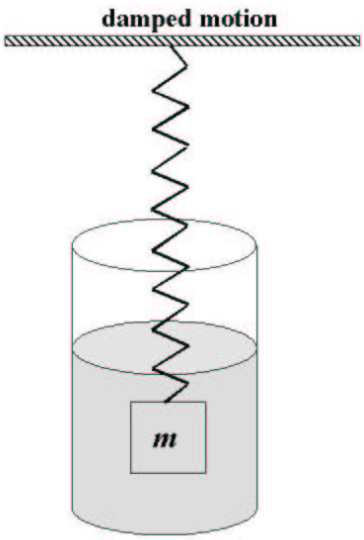

Figure 2

Vary all parameters of the applet and explore free damped motion.